Size-Space Bounds and Trade-offs for CDCL Proofs

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Joint work with Jan Johannsen,
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FLoC Workshop on Proof Complexity
Algorithms for SAT [1960s]

- State-of-the-art 1960s: DPLL

while not solved :
    unit_propagate()
    if conflict :
        backtrack()
    else :
        decide_variable_assignment()
Algorithms for SAT

- State-of-the-art 1960s: DPLL
- State-of-the-art now: Conflict Driven Clause Learning

while not solved :
    unit_propagate()
    if conflict :
        learn()
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Algorithms for SAT

- State-of-the-art 1960s: DPLL
- State-of-the-art now: Conflict Driven Clause Learning

while not solved :
    unit_propagate()
    if conflict :
        learn()
        maybe_restart()
        backjump()
    else :
        decide_variable_assignment()
What is the power of CDCL?

- Execution trace of CDCL $\rightarrow$ proof in resolution
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- Even true for most forms of preprocessing
What is the power of CDCL?

- Execution trace of CDCL $\rightarrow$ proof in subsystem of resolution
- Even true for most forms of preprocessing
- How strong a subsystem?
  - DPLL only as strong as tree-like
  - CDCL DAG-like, how much?
Known results

Line of research:
CDCL polynomially simulates resolution, but artificial models
[Beame, Kautz, Sabharwal ’04], [Van Gelder ’05],
[Hertel, Bacchus, Pitassi, Van Gelder ’08], [Buss, Hoffmann, Johannsen ’08]
Known results

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Under natural model (still some technical assumptions)
- CDCL p-simulates resolution [Pipatsrisawat, Darwiche '09]
- Randomized CDCL efficiently finds narrow proofs
  [Atserias, Fichte, Thurley '09]

Both results can be obtained from both papers
An even more faithful model?

Technical assumptions

- Unlimited memory
- Very frequent restarts
- Random decisions
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- Unlimited memory
- Very frequent restarts
- Random decisions

Relevant questions

- Can we model memory/space?
- How important are restarts?
- Performance of actual heuristic vs random decisions?
An even more refined model

Based on [Pipatsrisawat, Darwiche '09], [Atserias, Fichte, Thurley '09]; also ideas from [Buss, Hoffmann, Johannsen '08]
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- Proof as explicit resolution DAG + verification of CDCL-like
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- Proof as explicit resolution DAG + verification of CDCL-like

- Natural measure of space captures erasure of learnt clauses
A closer look at CDCL

```plaintext
while not solved :
    unit_propagate() // Unit resolution
    if conflict :
        learn()
        maybe_restart()
        backjump()
    else :
        assign_variable() // New decision
```

Data structures

- Branching sequence (assignments by decision or propagation)
- Clause database (list of learned clauses)
A closer look at CDCL

(Partial) input: \( z \land (x \lor y) \land (\overline{u} \lor \overline{w}) \land (u \lor \overline{w} \lor \overline{x} \lor y \lor \overline{z}) \)

while not solved :
    unit_propagate()
    if conflict :
        learn()
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        backjump()
    else :
        assign_variable()

Data structures
- Branching sequence
- Clause database: \( w \lor \overline{x} \lor y \lor \overline{z} \)
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(Partial) input: \[ z \land (x \lor y) \land (u \lor w) \land (u \lor w \lor \overline{x} \lor y \lor \overline{z}) \]

while not solved:
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Data structures

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(Partial) input: $z \land (x \lor y) \land (\overline{u} \lor \overline{w}) \land (u \lor w \lor \overline{x} \lor y \lor \overline{z})$

while not solved :
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Data structures

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\[
\begin{align*}
\text{while not solved :} & \\
& \text{unit\_propagate()} \\
& \text{if conflict :} \\
& \quad \text{learn()} \\
& \quad \text{maybe\_restart()} \\
& \text{else :} \\
& \quad \text{assign\_variable()}
\end{align*}
\]

\[
\begin{align*}
z = 1 & \quad z \\
y \overset{d}{=} 0 & \quad \text{(Decision)} \\
x = 1 & \quad x \lor y \\
w = 1 & \quad w \lor \overline{x} \lor y \lor \overline{z}
\end{align*}
\]

Data structures

- Branching sequence
- Clause database: \( w \lor \overline{x} \lor y \lor \overline{z} \)
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(Partial) input: \( z \land (x \lor y) \land (\overline{u} \lor \overline{w}) \land (u \lor \overline{w} \lor \overline{x} \lor y \lor \overline{z}) \)

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  u &= 0 \quad \overline{u} \lor w \\
\end{align*}
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(Partial) input: \( z \land (x \lor y) \land (\overline{u} \lor \overline{w}) \land (u \lor \overline{w} \lor \overline{x} \lor y \lor \overline{z}) \)

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Data structures

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while not solved:
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Next: \( y = 1 \) because of learned clause \( y \lor \overline{z} \)
Guaranteed by standard learning heuristic 1UIP
Proof system as annotated resolution

- Clauses as resolution DAG

\[
\begin{align*}
&y \lor \overline{z} \\
&x \lor y \\
&\overline{x} \lor y \lor \overline{z} \\
&w \lor \overline{x} \lor y \lor \overline{z} \\
&\overline{w} \lor \overline{x} \lor y \lor \overline{z} \\
&\overline{u} \lor \overline{w} \\
&u \lor \overline{w} \lor \overline{x} \lor y \lor \overline{z}
\end{align*}
\]
Proof system as annotated resolution

- Clauses as resolution DAG
- Grouped by sequences of input resolution
Proof system as annotated resolution

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- **Learned clauses** allowed in later steps
Proof system as annotated resolution

- Clauses as resolution DAG
- Grouped by sequences of input resolution
- **Learned clauses** allowed in later steps
- Branching sequence allows local checks

$$z=1 \mid y=d=0 \mid x=1$$
$$w=1 \mid u=0$$
Standard measures

- **Length** := resolution length (Here: 10)
- **# conflicts** := # input resolution sequences (Here: 4)
- **Space** := database size (Here: 2)
Some facts

Theorem

CDCL proof system polynomially simulates resolution length

By [Pipatsrisawat, Darwiche '09], [Atserias, Fichte, Thurley '09]
Some facts

**Theorem**

*CDCL proof system polynomially simulates resolution length*

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**Observation**

*CDCL proofs are valid resolution proofs*

Hence all lower bounds on length, space and trade-offs apply
Length and space upper bounds

Worst case for CDCL proof system same as resolution

**Proposition**

*Every formula has proofs in length $O(2^n)$ and space $O(n)$ simultaneously*

Not surprising, but also not immediate
Length and space upper bounds

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But can we simulate general resolution with respect to both length and space?
Length and space upper bounds

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**Proposition**

Every formula has proofs in length $O(2^n)$ and space $O(n)$ simultaneously

Not surprising, but also not immediate

But can we simulate general resolution with respect to both length and space? Regular resolution?
Length and space upper bounds

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Proposition

Every formula has proofs in length $O(2^n)$ and space $O(n)$ simultaneously

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But can we simulate general resolution with respect to both length and space? Regular resolution? Even tree-like resolution?
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**Proposition**

*Every formula has proofs in length \(O(2^n)\) and space \(O(n)\) simultaneously*

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Work in progress
Length-space trade-offs with restarts

Trade-offs from [Ben-Sasson, Nordström ’11] also hold with

Theorem

There exists a family of formulas such that:

1. There are short CDCL proofs
2. There are small CDCL proofs
3. Optimizing one measure blows up the other
Length-space trade-offs with restarts

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**Theorem**

There exists a family of formulas such that:

1. **There are short CDCL proofs**
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**Proof sketch.**

(3) immediate from resolution [BN’11]

Are there matching proofs in CDCL?
Length-space trade-offs with restarts

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There exists a family of formulas such that:

1. There are short CDCL proofs ✓
2. There are small CDCL proofs ✓
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**Proof sketch.**

(3) immediate from resolution [BN’11]
Are there matching proofs in CDCL? Yes
Simulate resolution clause by clause, restart at every clause
Space preserved
Standard 1UIP learning heuristic
Trade-offs without restarts

Theorem

There exists a specific family of formulas such that:

1. There are CDCL proofs in space \( s = O(1) \)
2. There are CDCL proofs in length \( L = O(n^2/s) \)
3. Every proof requires length \( L = \Omega(n^2/s^2) \)

Line of research investigating power of restarts

Upper bounds rely on restarts. Necessary?
Theorem

There exists a specific family of formulas such that:

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We craft explicit proofs without restarts for some families
Trade-offs without restarts

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Plausible for most results from [Ben-Sasson, Nordström '11] to follow
Technically involved, work in progress
More trade-offs?

Open: analogous results for the trade-offs in [Beame, Beck, Impagliazzo ’12] and [Beck, Nordström, Tang ’13]

- Conceivable for CDCL
- Less clear without restarts
Summary

- New CDCL proof system faithfully models:
  - Forgetting clauses
  - Restarts
  - Learning heuristics

Some upper bounds & trade-offs:

- All resolution lower bounds

Open Problems:

- Compare to resolution (general, regular, tree-like)
- Separate general resolution / CDCL with no restarts and 1UIP
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Thanks!
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