Cumulative Space in Black-White Pebbling and Resolution

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8th Innovations in Theoretical Computer Science
What is space?
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Usually: *maximal* space.
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Usually: *maximal* space.

[Alwen, Serbinenko ’15]: aggregate space over computation (*cumulative* space).
Resolution

Setup
Prove CNF formula unsatisfiable.
Present proof on board.

▶ Write down axiom clauses
▶ Infer new clauses
\[
\frac{C \lor x}{C \lor \neg x} \quad \frac{D \lor \neg x}{D \lor \neg x} \\
\hline
\frac{C \lor D}{C \lor D}
\]
▶ Erase clauses to save space

Goal: derive empty clause $\bot$

\[
F = \{x, \neg x \lor y, \neg y\}
\]
Resolution

Setup
Prove CNF formula unsatisfiable.

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- Write down axiom clauses

- Infer new clauses

\[
\begin{array}{c}
C \lor x \\
D \lor \overline{x}
\end{array}
\quad \Rightarrow \quad
\begin{array}{cc}
C \lor D
\end{array}
\]

- Erase clauses to save space

Goal: derive empty clause \( \bot \)

\[ F = \{ x, \overline{x} \lor y, \overline{y}\} \]
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▶ Erase clauses to save space

Goal: derive empty clause \( \perp \)

\[ F = \{ x, \overline{x} \lor y, \overline{y} \} \]
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- Write down axiom clauses
- Infer new clauses\[ \frac{C \lor x}{C \lor D} \quad \frac{D \lor \overline{x}}{C \lor D} \]
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Goal: derive empty clause \( \bot \)
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Prove CNF formula unsatisfiable.
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\[
\frac{C \lor x}{C \lor D} \quad \frac{D \lor \overline{x}}{C \lor D}
\]

▶ Erase clauses to save space

Goal: derive empty clause \( \bot \)

Questions
▶ How much time will this take? (Length)
▶ How large is the blackboard? (Space)
## Space

[Esteban, Torán ’99]

[Alekhnovich, Ben Sasson, Razborov, Wigderson ’00]

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Space of a proof: $Sp(\pi) := \max_t |\text{Clauses in } C_t| = 3$

Space of refuting a formula: $Sp(F \vdash \bot) := \min_{\pi: F \vdash \bot} Sp(\pi) \leq 3$
Space

[Esteban, Torán ’99]
[Alekhnovich, Ben Sasson, Razborov, Wigderson ’00]

\[
\begin{array}{ccccccc}
  x & x & x & \bar{x} \lor y & \bar{x} \lor y & y & \bar{y} \\
  x & \bar{x} \lor y & \bar{x} \lor y & y & y & \perp \\
\end{array}
\]

\[|C_1| = 1 \quad |C_2| = 2 \quad |C_3| = 3 \quad |C_4| = 2 \quad |C_5| = 1 \quad |C_6| = 2 \quad |C_7| = 3\]

Space of a proof: \(Sp(\pi) := \max_t |\text{Clauses in } C_t| = 3\)

Space of refuting a formula: \(Sp(F \vdash \bot) := \min_{\pi: F \vdash \bot} Sp(\pi) \leq 3\)

Alternative measures: \# literals, \# bits
Space

Bounds

Every formula $Sp = O(n)$
Exist formulas st $Sp = \Omega(n)$

[Esteban, Torán ’99], [Alekhnovich, Ben Sasson, Razborov, Wigderson ’00]
Space

Bounds
Every formula $Sp = O(n)$
Exist formulas st $Sp = \Omega(n)$
[Esteban, Torán ’99], [Alekhnovich, Ben Sasson, Razborov, Wigderson ’00]

Space vs length
Exist formulas st
- Exists proof with $Sp = O(n^{1/11})$
- Exists proof with $Len = O(n)$
- Every proof with $Sp < n^{2/11}$ requires $Len = \exp n^{\Omega(1)}$
[Ben Sasson, Nordström ’11]
Cumulative Space

Aggregate space over whole proof.

\[
\begin{array}{ccccccc}
  x & x &  & x & \overline{x} \lor y & \overline{x} \lor y & y & \overline{y} \\
  \overline{x} \lor y & \overline{x} \lor y & y & \overline{y} & \overline{y} & \perp \\
\end{array}
\]

\[|C_1| = 1 \quad |C_2| = 2 \quad |C_3| = 3 \quad |C_4| = 2 \quad |C_5| = 1 \quad |C_6| = 2 \quad |C_7| = 3\]

Cumulative space of a proof: \(\text{CumSp}(\pi) := \sum_t |\text{Clauses in } C_t| = 14\)
Cumulative space of refuting a formula:
\(\text{CumSp}(F \vdash \bot) := \min_{\pi: F \vdash \bot} \text{CumSp}(\pi) \leq 14\)
Cumulative Space

Observations

Every proof $\text{CumSp} \leq \text{Len} \cdot \text{Sp}$
Every formula $\text{Len} \leq 2^n$ and $\text{Sp} \leq n$
$\Rightarrow \text{CumSp} \leq n2^n$
**Cumulative Space**

**Observations**

Every proof $\text{CumSp} \leq \text{Len} \cdot \text{Sp}$

Every formula $\text{Len} \leq 2^n$ and $\text{Sp} \leq n$

$\Rightarrow \text{CumSp} \leq n2^n$

Every formula $\text{CumSp} \geq \text{Len}$

$\Rightarrow$ Most interesting if $\text{Len} = O(n)$. 
Cumulative Space

Observations

Every proof $\text{CumSp} \leq \text{Len} \cdot \text{Sp}$
Every formula $\text{Len} \leq 2^n$ and $\text{Sp} \leq n$
$\Rightarrow \text{CumSp} \leq n2^n$

Every formula $\text{CumSp} \geq \text{Len}$
$\Rightarrow$ Most interesting if $\text{Len} = O(n)$.

Every formula $\text{CumSp} \leq \text{Len}^2$. 
Cumulative Space

Observations

Every proof $\text{CumSp} \leq \text{Len} \cdot \text{Sp}$
Every formula $\text{Len} \leq 2^n$ and $\text{Sp} \leq n$
   $\Rightarrow \text{CumSp} \leq n2^n$

Every formula $\text{CumSp} \geq \text{Len}$
   $\Rightarrow$ Most interesting if $\text{Len} = O(n)$.

Every formula $\text{CumSp} \leq \text{Len}^2$.

Reaching space $s$ needs $s/2$ configurations of space $\geq s/2$
   $\Rightarrow$ Cumulative space $\Omega(s^2)$. 
Cumulative Space Bounds

How large can cumulative space be?

Every formula $\text{CumSp} = O(\text{Len}^2)$. Is this tight?
Cumulative Space Bounds

How large can cumulative space be?

Every formula $\text{CumSp} = O(\text{Len}^2)$. Is this tight?

Maximal space: $\text{Sp} = O(\text{Len})$ not tight.
Every formula $\text{Sp} = O(\text{Len} / \log \text{Len})$. [Hopcroft, Paul, Valiant ’75]
Cumulative Space Bounds

How large can cumulative space be?

Every formula $\text{CumSp} = O(\text{Len}^2)$. Is this tight?

Maximal space: $\text{Sp} = O(\text{Len})$ not tight.  
Every formula $\text{Sp} = O(\text{Len} / \log \text{Len})$. [Hopcroft, Paul, Valiant ’75]

**Theorem**

Exist formulas with $\text{Len} = O(n)$ and $\text{CumSp} = \Omega(n^2)$.
Maximal vs Cumulative Space

Large space $\iff$ large cumulative space?

$\Rightarrow$ Yes
Every formula $\text{CumSp} = \Omega(Sp^2)$.
Maximal vs Cumulative Space

Large space $\Leftrightarrow$ large cumulative space?

$\Rightarrow$ Yes
Every formula $\text{CumSp} = \Omega(Sp^2)$.

$\Leftarrow$ No

Theorem

Exist formulas with $Sp = O(\log n)$ but $\text{CumSp} = \Omega(n^2 / \log n)$.
Length vs Cumulative Space

How often do we need maximum space in a trade-off?

Theorem [Ben Sasson, Nordström ’11]

Exist formulas st for any $s = O(\sqrt{n})$

- Exists proof with $Sp = O(s)$ and $Len = O(n^2/s^2)$
  - Exists proof with $Sp = O(1)$
  - Exists proof with $Len = O(n)$
- Every proof in space $O(s)$ needs $Len = \Omega(n^2/s^2)$
Length vs Cumulative Space

How often do we need maximum space in a trade-off?

Theorem [Ben Sasson, Nordström '11]

Exist formulas st for any $s = O(\sqrt{n})$

- Exists proof with $Sp = O(s)$ and $Len = O(n^2/s^2)$
  - Exists proof with $Sp = O(1)$
  - Exists proof with $Len = O(n)$

- Every proof in space $O(s)$ needs $Len \cdot Sp = \Omega(n^2/s)$
Length vs Cumulative Space

How often do we need maximum space in a trade-off?

Theorem

Exist formulas st for any $s = O(\sqrt{n})$

- Exists proof with $Sp = O(s)$ and $Len = O(n^2/s^2)$
  - Exists proof with $Sp = O(1)$
  - Exists proof with $Len = O(n)$
- Every proof in space $O(s)$ needs $CumSp = \Omega(n^2/s)$

Corollary

- Every proof in space $O(s)$ and length $O(n^2/s^2)$ needs $\Omega(n^2/s^2)$ configurations with space $\Omega(s)$
Parallel Resolution

Parallel resolution: allow many steps at once.

Automatic $\text{CumSp} = \Omega(Sp^2)$ lower bound no longer holds.
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Parallel Inference
Previous results hold even allowing parallel inference.
Parallel Resolution

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Automatic $\text{CumSp} = \Omega(Sp^2)$ lower bound no longer holds.

Parallel Inference
Previous results hold even allowing parallel inference.

Fully Parallel Resolution
Very powerful model: can prove any formula in 2 steps.
Lower bounds with limited space.
Techniques

Pebble games

- Simple computational model to measure space.
- Prove lower bounds in pebble game.
- Translate to resolution.
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Pebble games

- Simple computational model to measure space.
- Prove lower bounds in pebble game
- Translate to resolution

Lemma

Resolution proof of $F(G)$ in length $L$, space $s$, cumulative space $c$. Then pebbling of $G$ in time $L$, space $s$, cumulative space $c$.

Even if parallel inference steps.
Techniques

Pebble games

- Simple computational model to measure space.
- Prove lower bounds in pebble game
- Translate to resolution

Lemma

Resolution proof of $F(G)$ in length $L$, space $s$, cumulative space $c$. Then pebbling of $G$ in time $L$, space $s$, cumulative space $c$.

Even if parallel inference steps.

- [Alwen, Serbinenko ’15]: Translate computation to black pebbling strategy.
- Proofs are non-deterministic: translate proof to black-white pebbling.
Take Home

Recap

- Introduced cumulative space measure in proof complexity.

Open problems

- Study cumulative space in other areas.
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Open problems

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Thanks!