Smooth Games, Price of Anarchy and Composability of Auctions - a Quick Tutorial

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Abstract

In this tutorial we review the concept of smooth games as introduced by Roughgarden in [1]. We show how to upper bound the Price of Anrchy of different games using this general tool. Then along the lines of [2], we extend the idea of smooth games to smooth mechanisms and weakly smooth mechanisms and prove that many of the common auctions fall under this category. We then show how efficiency results of these mechanisms in a Bayes-Nash equilibria may be derived from smoothness arguments. Then we prove that simultaneous composition of smooth mechanism preserves the smoothness property. Hence all the previous results about efficiency of smooth mechanisms carry over to simultaneous compositions.

I. Introduction

Selfish behavior by individual decision makers generally leads to an inefficient result, an outcome which could be improved upon given the individuals coordinate their actions together. The Price of Anarchy (POA) measures the sub-optimality caused by selfish behaviors. Given a game, a notion of an equilibrium (such as pure Nash equilibria), and a non-negative objective function (such as the sum of players' costs), the POA of the game is defined as the ratio between the largest cost of an equilibrium and the cost of an optimal outcome. We compute POA using a smoothness argument in this tutorial. We also consider extension of the smoothness framework in the setting of mechanism design and show that a number of popular auctions fall in this category. It follows that smoothness of auctions ensures efficiency for Bayesian Nash Equilibrium of the underlying game. Finally, we prove that simultaneous compositions of smooth mechanisms are smooth. This result is particularly important for online market settings where each bidders participate in a multitude of auctions simultaneously (e.g. in eBay). Hence the above result implies that smooth auctions with good efficiency bounds may be run simultaneously without compromising their overall efficiency guarantee.

II. Smooth Games

The concept of smooth games was first introduced by Roughgarden in [1]. He defined it first in the context of cost-minimization games. A cost-minimization game is defined by a 3-tuple $(\mathcal{N}, \mathcal{S}, \mathcal{C})$, where $\mathcal{N}$ is the set of players, $\mathcal{A} = \prod_i \mathcal{A}_i$, where $\mathcal{A}_i$ is the set of strategies available for the i^{th} player and $\mathcal{C} = \prod_i C_i$, where $C_i : \mathcal{S} \to \mathbb{R}_+$ is the cost function for the i^{th} player. The social-cost for the game for the strategy-vector $\boldsymbol{s} \in \mathcal{S}$ is given by the separable function

$$C(\boldsymbol{s}) = \sum_{i \in \mathcal{N}} C_i(\boldsymbol{s}) \tag{1}$$

A. Smooth Games [1]

Formally, Roughgarden defined a cost-minimization game to be (λ, μ) smooth if the following holds for any strategy-pairs $(\boldsymbol{s}^*, \boldsymbol{s})$:

$$\sum_i C_i(s_i^*, \boldsymbol{s}_{-i}) \leq \lambda C(\boldsymbol{s}^*) + \mu C(\boldsymbol{s}) \tag{2}$$

Intuitively, given two strategies $\boldsymbol{s}$ and $\boldsymbol{s}^*$, a game is called smooth if the sum of the costs of the players due to unilateral deviation from $\boldsymbol{s}$ to $\boldsymbol{s}^*$ can be upper bounded by a linear combination of the social-costs at strategies $\boldsymbol{s}$ and $\boldsymbol{s}^*$.

*Definition 1 (**Price of Anarchy**):* For a given cost-minimization game, let $\boldsymbol{s}^*$ be an optimal action-vector for the player and let $\hat{\boldsymbol{s}}$ be any Nash-equilibrium. The Price-of-Anarchy (PoA) is defined as the supremum of the ratio $\frac{C(\hat{\boldsymbol{s}})}{C(\boldsymbol{s})}$, where the supremum is taken over all Nash-Equilibriums of the underlying game.

The above definition of smoothness immediately yields the following upper-bound on the Price-of-anarchy :

Theorem 1: If a cost-minimization game is (λ, μ) smooth with $\lambda > 0$ and $\mu < 1$, then Price-of-anarchy (PoA) is upper-bounded by $\frac{\lambda}{1-\mu}$.

Proof: Let $\boldsymbol{s}^*$ be a socially-optimal strategy vector, i.e. :

$$\boldsymbol{s}^* \in \arg\min_{\boldsymbol{s} \in \mathcal{S}} C(\boldsymbol{s}) \tag{3}$$

Also, let $\hat{\boldsymbol{s}}$ be a Nash-equilibrium of the game. Then by definition of Nash-equilibrium, we have

$$\begin{aligned} C(\hat{\boldsymbol{s}}) &\equiv \sum_i C_i(\hat{s}_i, \hat{\boldsymbol{s}}_{-i}) & (4)\\ &\leq \sum_i C_i(s_i^*, \hat{\boldsymbol{s}}_{-i}) & (5)\\ &\leq \lambda C(\boldsymbol{s}^*) + \mu C(\hat{\boldsymbol{s}}) & (6) \end{aligned}$$

Here Eqn. (4) follows from the definition of social cost function, Eqn. (5) follows from the definition of Nash equilibrium and Eqn. (6) follows from the smoothness property of the game. The above inequality implies

$$C(\hat{\boldsymbol{s}}) \leq \frac{\lambda}{1-\mu} C(\boldsymbol{s})$$

This establishes the result. ■

B. An Example of Computation of PoA using Smoothness Arguments

1) Congestion Games with Affine Cost Functions: A congestion game is a cost-minimization game defined by a ground set E f resources, a set k of players with strategy sets $S_1, S_2, \cdots, S_k \subset 2^E$. The abstract framework may be understood as follows, Consider a graph $\mathcal{G}(V, E)$. Each player i is associated with a source-destination pair (s_i, t_i) that they wish to travel. The strategy set of each player is simply the set of distinct paths from his source to the destination. When x_e of the users avail the edge $e \in E$, they incur a cost of $c_e(x_e) = a_e x_e + b_e$. Hence, given a strategy profile $\boldsymbol{s} = \{s_1, s_2, \ldots, s_k\}$, the load induced on the edge e is given by $x_e = |i : e \in s_i|$. The cost incurred by the player i is simply $C_i(\boldsymbol{s}) = \sum_{e \in S_i} c_e(x_e)$ and the total cost of the game is $C(\boldsymbol{s}) = \sum_i C_i(\boldsymbol{s})$. Interchanging the order of summations, it follows that

$$C(\boldsymbol{s}) = \sum_{e \in E} x_e c_e(x_e) \tag{7}$$

We have the following theorem on the PoA of Congestion games :

Theorem 2: The Price of Anarchy of congestion games is at most $\frac{5}{2}$.

Proof: Let the strategy set (possibly randomized) $\boldsymbol{s}$ be a Nash Equilibrium and $\boldsymbol{s}^*$ is the socially optimal strategy. Observe that, in the modified strategy $(s_i^*, \boldsymbol{s}_{-i})$, the number of players using resource e is at most one more than that in $\boldsymbol{s}$ and this resource contributes to precisely x_e^* terms of the form $C_i(s_i^*, \boldsymbol{s}_{-i})$. Hence

$$\sum_{i=1}^{k} C_i(s_i^*, \boldsymbol{s}_{-i}) \leq \sum_{e \in E} (a_e(x_e + 1) + b_e) x_e^* \tag{8}$$

To proceed further, we need the following lemma :

Lemma 1: For all non-negative integers y, z, the following holds:

$$y(z+1) \leq \frac{5}{3} y^2 + \frac{1}{3} z^2 \tag{9}$$

Proof:

a) Case 1: $y = 0$: Trivial.

b) Case 2: $y = 1$: In this case, we need to show that for all non-negative integers z, we have

$$z + 1 \leq \frac{5}{3} + \frac{1}{3} z^2 \tag{10}$$

i.e.,

$$(z-1)(z-2) \geq 0 \tag{11}$$

which is clearly true for all $z \in \mathbb{Z}_+$.

c) Case 3:$y \geq 2$: We need to show that :

$$z^2 - 3yz + 5y^2 - 3y \geq 0 \quad (12)$$

Treating the above as a quadratic in z, it is enough to show that the discriminant is negative, i.e.

$$9y^2 - 4(5y^2 - 3y) < 0 \quad (13)$$

which is true if $y > \frac{12}{11}$, i.e. if $y \geq 2$. ■

Now we return to the proof of the main result. Using the above lemma, from Eqn. (8) we have

$$\sum_{i=1}^{k} C_i(s_i^*, \boldsymbol{s}_{-i}) \leq \frac{5}{3}\sum_{e\in E}(a_e x_e^* + b_e)x_e^* + \frac{1}{3}\sum_{e\in E}(a_e x_e + b_e)x_e \quad (14)$$

$$= \frac{5}{3}C(\boldsymbol{s}^*) + \frac{1}{3}C(\boldsymbol{s}) \quad (15)$$

Finally, appealing to theorem (1), we obtain the result. ■

III. Smooth Mechansims

Motivated by smooth games defined by Roughgarden, Syrgkanis and Tardos [2] generalized the idea to mechanism design settings. They investigated the setting of simultaneous mechanisms and showed that if the component mechanisms satisfies smoothness conditions, then the overall mechanism also satisfies smoothness conditions and hence, proving a smoothness bound on individual mechanism yields an automatic smoothness bound on the composed mechanism.

2) Mechanisms with Quasilinear Preferences : A mechanism design setting consists of a set of n players and a set of outcomes $\mathcal{X} \subset \times_i \mathcal{X}_i$, where $\mathcal{X}_i$ is the set of allocations for player i. Each player has a valuation $v_i : \mathcal{X}_i \to \mathbb{R}_+$. Given an allocation $x_i \in \mathcal{X}_i$ and a payment $p_i \in \mathbb{R}_+$, the utility of the player i is simply given by

$$u_i(x_i, p_i) = v_i(x_i) - p_i \quad (16)$$

Given an outcome space $\mathcal{X}$, a mechanism $\mathcal{M}$ is a tuple $(\mathcal{S}, \mathcal{X}, P)$, where $\mathcal{S} = \prod_i S_i$ is the strategy space of the players, $X : \mathcal{S} \to \mathcal{X}$ is the allocation function and $\mathcal{P} : \mathcal{S} \to \mathbb{R}_n^+$ is the payment rule.

Definition 2: (Smooth Mechanism) A mechanism $\mathcal{M}$ is (λ, μ) smooth, if for any valuation profile $v \in \times\mathcal{V}_i$ and any action profile $\boldsymbol{a}$, there exists a randomized action $a_i^*(v, a_i)$ for each player i such that :

$$\sum_i u_i(a_i^*(v, a_i), a_{-i}) \geq \lambda \textsc{Opt}(v) - \mu \sum_i P_i(a) \quad (17)$$

Similar to theorem (1), a smooth mechanism immediately implies that social welfare at any Correlated equilibrium of the game is at least a constant fraction of the optimal social welfare.

Theorem 3: If a mechanism is (λ, μ) smooth and satisfies IR and NPT properties then the expected social welfare at any Correlated Equilibrium of the game is at least $\frac{\lambda}{\max\{\mu,1\}}$ of the optimal social welfare.

Proof: Assume that the (possibly randomized) action-profile $\boldsymbol{a}$ constitutes a correlated equilibrium of the game. Then we have, by the definition of correlated equilibrium, for each player i and for any other action profile a_i^*

$$u_i(a_i, a_{-i}) \geq u_i(a_i^*, a_{-i}) \quad (18)$$

Thus,

$$W(\boldsymbol{a}) = \sum_i (v_i(X_i(\boldsymbol{a})) \quad (19)$$

$$= \sum_i u_i(a_i, a_{-i}) + \sum_i P_i(a) \quad (20)$$

$$\geq \sum_i u_i(a_i^*(v, a_i), a_{-i}) + \sum_i P_i(a) \quad (21)$$

$$\geq \lambda \textsc{Opt}(v) + (1 - \mu)\sum_i P_i(a) \quad (22)$$

Where Eqn. (20) follows from the definition of quasilinear utility, Eqn. (21) follows from Eqn. (18) and Eqn. (22) follows from the smoothness assumption of the mechanism. If $\mu < 1$, the result follows because of NPT property of the mechanism ($P_i(a) \geq 0$).

On the other hand, if $\mu > 1$, since at any correlated equilibrium, every player derives a non-negative utility (due to the mechanism being IR). This implies $v_i(X_i(\boldsymbol{a})) \geq P_i(\boldsymbol{a})$, i.e. $W(\boldsymbol{a}) \geq \sum_i P_i(\boldsymbol{a})$. Hence, from Eqn. (22), we have

$$W(\boldsymbol{a}) \geq \lambda \text{OPT}(v) + (1-\mu)W(\boldsymbol{a}) \tag{23}$$

Thus, we have

$$W(\boldsymbol{a}) \geq \frac{\lambda}{\mu}\text{OPT}(v) \tag{24}$$

■

This proves the result.

A. Smoothness of Popular Auctions

In this section we show that two popular auctions, namely, First Price and All-Pay auctions are smooth. The derivation consists of deriving a smoothness inequality of the form (17), by considering a hypothetical deviation from any bidding strategy $\boldsymbol{b}$.

1) First Price Auction:

Description: In first price auction there is one item to be auctioned off. Each bidder i has a private value v_i and submits a sealed-bid b_i. We allocate the the item to the bidder with highest bid and pays an amount equal to his bid. The rest of the bidder pays zero.

Theorem 4: The First-Price auction is $(1-1/e, 1)$ smooth.

Proof: Consider a valuation profile $\boldsymbol{v}$ and bidding profile $\boldsymbol{b}$. Then we have $\text{OPT}(\boldsymbol{v}) = \max_i v_i$ and $\sum_i P_i(a) = \max b_i$. Now consider the following deviation from the bidding profile $\boldsymbol{b}$: The bidder with the highest valuation $\max_i v_i \equiv v_{\max}$ submits a bid randomly sampled from the distribution $f(b^*) = \frac{1}{v_{\max} - b^*}$ and the support $[0, (1-1/e)v_{\max}]$ [1] and the rest of the bidders submit zero.

Let us denote the highest bidder by i^*. When the rest of the bidders play b_{-i} and he samples $b^* = b$, he derives the following utility :

$$u_{i^*}(b, b_{-i}) = \begin{cases} v_{\max} - b & \text{if } b > \max_{k \neq i^*} b_k \\ 0 & \text{o.w.} \end{cases} \tag{25}$$

Thus the expected utility of the player i^* is simply

$$\begin{aligned} u_i(b^*, b_{-i}) &= \int_{\max_{k\neq i^*} b_k}^{(1-1/e)v_{\max}} \frac{v_{\max} - b}{v_{\max} - b} db \\ &\geq (1-1/e)v_{\max} - \max_k b_k \\ &= (1-1/e)\text{OPT}(v) - \sum_i P_i(\boldsymbol{a}) \end{aligned} \tag{26}$$

Since, utility of the rest of the bidders are non-negative, the above inequality shows that first price auction is $(1-1/e, 1)$ smooth. ■

2) All-Pay Auction:

Description: Like the first price auction, in an All-Pay auction there is only one item to be auctioned off and each bidder i has private value v_i for the item. Every bidder submits a sealed bid b_i. The item is allocated to the highest bidder, but unlike the first price auction, every bidder must pay his bid, irrespective of whether he is allocated the item or not. Note that the All-Pay auction is not IR.

Theorem 5: The All-Pay auction is an $(\frac{1}{2}, 1)$ smooth mechanism.

Proof: For a submitted bid-profile $\boldsymbol{b}$, we consider similar deviation as in the First-Pay auction. This time, the highest valued bidder submits a bid b^* sampled uniformly at random from the interval $[0, v_{\max}]$ and all bidders submit zero. We have $\text{OPT}(v) = v_{\max}$ and $\sum_i P_i(a) = \sum_i b_i$. The utility of the highest valued bidder, denoted by i^*, is written as follows:

$$u_{i^*}(b, b_{-i}) = \begin{cases} v_{\max} - b & \text{if } b > \max_{k \neq i^*} b_k \\ 0 & \text{o.w.} \end{cases} \tag{27}$$

[1] It can be easily checked that it is a valid probability distribution.

Thus expected utility of the player i^* is simply given by

$$\begin{aligned} u_i(b^*, b_{-i}) &= \int_{\max_{k\neq i^*} b_k}^{v_{\max}} \frac{1}{v_{\max}}(v_{\max} - b)db \\ &\geq v_{\max} - \max_{k\neq i^*} b_k - \frac{1}{2}v_{\max} \\ &\geq \frac{1}{2}\textsc{Opt}(v) - \sum_i P_i(b) \end{aligned} \tag{28}$$

The above shows that All-Pay auction is $(\frac{1}{2}, 1)$ smooth. ■

3) First Price Public Project Auction:

Description: There are n bidders and m public projects. Each player i has value v_{ij} if project j is implemented. The bidders submit their bids b_{ij} and the mechanism chooses the project j which maximizes $\sum_i b_{ij}$. If project j^* is selected, player i pays b_{ij^*}. Hence $\sum_i P_i(b) = \sum_i b_{ij^*}$

Theorem 6: The First Price public project auction is $\left(\frac{1-e^{-n}}{n}, 1\right)$ smooth.

Proof: Let j^* be the optimal project for a given valuation profile $\boldsymbol{v} = \{v_{ij}\}$. Suppose each player i deviates to bidding $b_i^* \in [0, (1-e^{-n})v_{ij^*}]$ with probability density function $f(t) = \frac{1/n}{v_{ij^*} - b_i^*}$ to the project j^* and zero otherwise. Let $j(b)$ be the project chosen according to the submitted bid b. Clearly, if $b_i^* > \sum_i b_{ij(b)}$, then the project j^* would be chosen instead and player i would derive an utility of $v_{ij^*} - b_i^*$. Thus

$$u_i(b_i^*, b_{-i}) \geq \int_{\sum_i b_{ij(b)}}^{(1-e^{-n})v_{ij^*}} (v_{ij^*} - b_i^*)\frac{1/n}{v_{ij^*} - b_i^*}db_i^* \tag{29}$$

$$\geq \frac{1}{n}(1-e^{-n})v_{ij^*} - \frac{1}{n}\sum_{i=1}^{n} b_{ij(b)} \tag{30}$$

Summing over all bidders i, we obtain

$$\sum_{i=1}^{n} u_i(b_i^*, b_{-i}) \geq \frac{1}{n}(1-e^{-n})\textsc{Opt}(v) - \sum_i P_i(b) \tag{31}$$

This proves the result. ■

B. Weak Smoothness

In this section we give a generalization of the smoothness framework to capture mechanisms that produce high-efficiency under a no-overbidding refinement. The most prominent example among these mechanisms is the celebrated Second Price Auction. In the second-price auction the bid of a player is his maximum willingness to pay when he wins. We start with the following definitions :

*Definition 3 (**Willingness to Pay**):* Given a mechanism $(\mathcal{A}, \mathcal{X}, \mathcal{P})$ a player's maximum willingness-to-pay for an allocation x_i when using strategy a_i is defined as the maximum he could ever pay conditional on allocation x_i :

$$B_i(a_i, x_i) = \max_{a_{-i}:X_i(a)=x_i} P_i(a) \tag{32}$$

*Definition 4 (**Weakly Smooth Mechanism**):* A mechanism is weakly (λ, μ_1, μ_2)-smooth for $\lambda, \mu_1, \mu_2 \geq 0$ if for any type profile $v \in \times_i \mathcal{V}_i$ and for any action profile a there exists a randomized auction $a_i^*(v, a_i)$ for each player i, s.t.

$$\sum_i u_i^{v_i}(\boldsymbol{a}_i^*(v, a_i), a_{-i}) \geq \lambda \textsc{Opt}(\boldsymbol{v}) - \mu_1 \sum_i P_i(a) - \mu_2 \sum_i B_i(a_i, X_i(a)) \tag{33}$$

*Definition 5 (**No-overbidding**):* A randomized stategy profile $\boldsymbol{a}$ satisfies the no-overbidding assumption if:

$$\mathbb{E}[B_i(a_i, X_i(a))] \leq \mathbb{E}[v_i(X_i(a))] \tag{34}$$

Under the assumption of no-overbidding and Weakly smooth mechanism, we have an efficiency result similar to (3).

Theorem 7: If a mechanism is weakly (λ, μ_1, μ_2)-smooth then any Correlated Equilibrium in the full information setting and any mixed Bayes-Nash Equilibrium in the Bayesian Setting that satisfies the no-overbidding assumption achieves efficiency at least $\frac{\lambda}{\mu_2 + \max\{1, \mu_1\}}$ of the expected optimal.

C. Examples of Weakly Smooth Mechanisms

1) γ-hybrid auction: In γ-hybrid auction, bidders submit their bids for procurement of a single item. The bidder submitting the highest bid wins and pays his bid with probability γ and the second highest bid with probability $(1-\gamma)$.

Theorem 8: The γ-hybrid auction is $(\gamma(1-1/e)+(1-\gamma)^2, 1, (1-\gamma)^2)$ smooth.

Proof: Consider a valuation profile $\boldsymbol{v}$ and a bid profile $\boldsymbol{b}$. Let $v_{\max}$ and $b_{\max}$ be the highest value of the bidders and and their bids respectively. Consider the following deviation to the bid profile $\boldsymbol{b}$. The highest value bidder bids $v_{\max}$ with probability $1-\gamma$, and with bids $b^* \in [0,(1-1/e)v_{\max}]$ with density $f(b^*) = \frac{1}{v_{\max}-b^*}$. All remaning bidders deviate to bidding zero in the modified profile. Clearly, if $b^* > b_{\max}$, the highest valued bidder wins. Hence his utility may be written as follows:

Case 1: i^* bids his true value
Then if $v_{\max} \geq b_{\max}$ he wins the item and gets an utility of

$$u_{i^*}(b_i^*, \boldsymbol{b}_{-i}) = v_{\max} - \gamma v_{\max} - (1-\gamma)b_{\max} = (1-\gamma)(v_{\max} - b_{\max}) \tag{35}$$

If $v_{\max} < b_{\max}$, he loses and gets zero utility. Hence his utility in this case is at least

$$(1-\gamma)(v_{\max} - b_{\max}) \tag{36}$$

Case 2: i^* bids according to the random strategy

$$\begin{aligned}
u_{i^*}(b^*, \boldsymbol{b}_{-i^*}) &\geq \int_{b_{\max}}^{(1-1/e)v_{\max}} \big(v_{\max} - \gamma b^* - (1-\gamma)b_{\max}\big)\frac{1}{v_{\max}-b^*}db^* &(37)\\
&\geq \int_{b_{\max}}^{(1-1/e)v_{\max}} \big(v_{\max} - b^*\big)\frac{1}{v_{\max}-b^*}db^* &(38)\\
&= (1-1/e)v_{\max} - b_{\max} &(39)
\end{aligned}$$

Thus his overall utility is at least

$$\gamma\big((1-1/e)v_{\max} - b_{\max}\big) + (1-\gamma)\big((1-\gamma)(v_{\max} - b_{\max})\big)The \tag{40}$$

The lemma follows by just observing that the payment under bid profile $\boldsymbol{b}$ is just $\gamma b_{\max}$. ■

*D. **The Composition Framework***

Mechanisms rarely run in isolation but rather, several mechanisms take place simultaneously and players typically have valuations that are functions on the outcomes of different mechanisms.

Consider the following general setting: there are n bidders and m mechanisms. Each mechanism $\mathcal{M}_j$ has its own outcome space $\mathcal{X}^j$ and consists of a tuple $(\mathcal{A}^j, X^j, P^j)$. We assume that a player has a valuation over vectors of outcomes from different mechanisms : $v_i : \times_i \mathcal{X}_i \to \mathbb{R}_+$. The player i's overall utility for the outcome $\boldsymbol{x}_i = (x_i^1, x_i^2, \ldots, x_i^m)$ and payment $\boldsymbol{p}_i = (p_i^1, p_i^2, \ldots, p_i^m)$ is given by the quasilinear function

$$u_i(x_i, p_i) = v_i(\boldsymbol{x}_i(a)) - \sum_j p_i^j \tag{41}$$

The simultaneous composition of such mechanisms may be viewed as a global mechanism $\mathcal{M} = (\mathcal{A}, X, P)$, where $\mathcal{A}_i = \times_j \mathcal{A}_i^j, \mathcal{X} = \times_j \mathcal{X}^j$ and $P_i(a) = \sum_j p_i^j$. The social-welfare of a strategy-profile $\boldsymbol{a}$ has the following separable form :

$$W(\boldsymbol{a}) = \sum_i v_i(x_i(\boldsymbol{a})) \tag{42}$$

For any valuation profile v, there exists an allocation $x^*(v)$ which maximizes $\sum_i v_i(x_i)$ over all $\mathcal{X}$. We denote the resulting optimal social welfare by $\textsc{Opt}(v)$.

We restrict ourselves to a class of valuation functions known as **XOS** valuation defined below.

*Definition 6 (**XOS Valuation**):* A valuation is XOS if there exist a set $\mathcal{L}$ of additive valuations $v_j^l(x_j)$, such that

$$v(\boldsymbol{x}) = \max_{l\in\mathcal{L}} \sum_j v_j^l(x_j)$$

We have the following theorem regarding the smoothness of composed mechanisms

*Theorem 9 (**Simultaneous Composition**):* Consider the simultaneous composition of m mechanisms. Suppose that each mechanism $\mathcal{M}_j$ is (λ, μ) smooth when the mechanism restricted valuations of the players come from a class $(\mathcal{V}_i^j)_{i\in[n]}$. If the valuation $v_i : \mathcal{X}_i \to \mathbb{R}^+$ of each player across mechanisms is XOS and can be expressed by component valuations v_{ij}^l, then the global mechanism is also (λ, μ)-smooth.

Proof: Consider a valuation profile $\boldsymbol{v}$ and an action profile $\boldsymbol{a}$. Let $\boldsymbol{x}^*$ be the optimal allocation for the profile $\boldsymbol{v}$. Let v_{ij}^* be the representative additive valuation for x_i^*, i.e.

$$v_i(x_i^*) = \sum_j v_{ij}^*(x_{ij}^*) \tag{43}$$

Also, from the definition of XOS valuation, for any $x_i \in \mathcal{X}_i$

$$v_i(x_i) \geq \sum_j v_{ij}^*(x_i^j) \tag{44}$$

To prove the theorem, we will show that there exists a deviation $\boldsymbol{a}_i^* = \boldsymbol{a}_i^*(\boldsymbol{v}, a_i)$ of the global mechanism such that

$$\sum_i u_i^{v_i}(\boldsymbol{a}_i^*, a_{-i}) \geq \lambda \sum_i v_i(x_i^*) - \mu \sum_i P_i(a) \tag{45}$$

To define such a deviation we use the fact that each mechanism $\mathcal{M}_j$ is (λ, μ) smooth. Suppose that we run mechanism $\mathcal{M}_j$ and each player has valuation v_{ij}^* on $\mathcal{X}_i^j$. By the smoothness property of the mechanism $\mathcal{M}_j$, for any action profile $\boldsymbol{a}^j$, there exists a randomized action $\boldsymbol{a}_{ij}^* = \boldsymbol{a}_i^*(v_j^*, a_i^j)$ such that the sum of the utilities of the agents when each agent unilaterally deviates to it, is at least $\lambda \sum_i v_{ij}^*(x_{ij}^*) - \mu \sum_i P_i^j(a^j)$.
For the global mechanism, we consider a randomized deviation $\boldsymbol{a}_i^* = \boldsymbol{a}_i^*(v, a_i)$, that consists of independent randomized deviations $a_{ij}^* = a_{ij}^*(v_j^*, a_i^j)$. By the properties of the representative additive valuations, we have

$$v_i(X_i(a_i^*, a_{-i}) \geq \sum_j v_{ij}^*(X_i^j(a_{ij}^*, a_{-i}^j) \tag{46}$$

Hence,

$$\sum_i u_i^{v_i}(\boldsymbol{a}_i^*, a_{-i}) \geq \sum_{j,i} \mathbb{E}[v_{ij}^*(X_i^j(a_{ij}^*, a_{-i}^j) - P_i^j(\boldsymbol{a}_{ij}^*, a_{-i}^j)] \tag{47}$$

By smoothness of mechanism $\mathcal{M}_j$:

$$\sum_i u_i^{v_i}(\boldsymbol{a}_i^*, a_{-i}) \quad \geq \quad \sum_j (\lambda \sum_i v_{ij}^*(x_{ij}^*) - \mu \sum_i P_i^j(a_j)) \tag{48}$$

$$= \quad \lambda \sum_i v_i(x_i^*) - \mu \sum_i P_i(a) \tag{49}$$

■

Corollary 1: Simultaneous compositions of First Price Auctions, All-Pay auctions and Public project auctions are $[(1 - 1/e), 1], [1/2, 1], [(1 - e^{-n})/n, 1])$-smooth respectively.

IV. Conclusion

In this tutorial, we have introduced the basic idea of smoothness of games and extension of this concept to mechanism design. We showed that smoothness is a general tool which may be used effectively in determining the Price of Anarchy of games and efficiency of mechanisms. We also showed that simultaneous composition of (λ, μ) smooth mechanisms are (λ, μ)-smooth. Future direction would be to analyze PoA of more games using the smoothness framework and obtain better bounds of efficiency of new mechanisms.

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