Resources for Quantum Technologies

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Abstract
Development of technologies based on principles of quantum mechanics is one of the challenging tasks around the world. For this purpose, identifying proper quantum mechanical state which can be used to build quantum technology is important. I will review the characteristic of such resources, possible generation and their applications in quantum information processing tasks.

Introduction
Based on the performance, technologies exploiting quantum mechanical principles can broadly be classified into two categories — communication protocols in which capacities in sending information can be increased by using quantum states and quantum operations, and computational schemes which can complete certain assignments in polynomial time while classical computational schemes cannot [1–3]. The former scenario includes quantum state transfer, classical information transmission, cryptographic schemes, while the latter one considers solving mathematical problems like finding prime factors of an integer.

A set of states without which quantum protocols cannot be accomplished is called quantum resources and is important to identify. At the same time, it is also interesting to find out the specific characteristics of these states, which lead to the improvement. For example, in communication schemes, the quantum property called entanglement [4] shared between the parties involved are shown to be essential. Other non-classical resources, independent of entanglement, have also been found and have attracted lots of attention in recent times [5].

Apart from theoretical predictions establishing superiority of quantum mechanics towards building technologies, tremendous experimental achievements have been reported, especially in last ten years [6, 7]. For example, polarization of photons [8] and internal levels of ions [9, 10] can act as two dimensional quantum systems, namely qubit [11]. These degrees of freedom of a particular physical system can be exploited to produce entangled states. Currently, two-party highly entangled states are routinely prepared in laboratories with these physical systems. Other potential candidate for implementing quantum protocols are superconducting qubits [12], atoms in a cavity [13] or in an optical lattice [14, 15], nuclear magnetic resonances [16].

In this article, we will only discuss the concept of entanglement including detection as well as quantification methods. We show a possible way to generate entangled states. We then describe a communication protocol where entangled state are essential ingredient.

Resource States
Depending on quantum information processing tasks, classification of resource states has been made [4, 5]. In this section, we will define one of the most important resource in quantum information science, namely entanglement [4]. We first define entangled states shared by at least two spatially separated observers. We will then go beyond a two-party scenario. Finally, we will show a way to identify and quantify entanglement.

A. Definition of Entanglement
In entanglement paradigm, useless states are the unentangled (also called the product or separable states) which can be prepared by two parties using local operations and classical communication (LOCC). In this scenario, LOCC can be called free operations. More specifically, two parties, Alia and Brato, can prepare these useless states even when they are situated in two distant locations. In case of pure states, product states can mathematically be written as

\[ |\psi_{AB}⟩ = |\psi_A⟩ \otimes |\psi_B⟩, \]

while entangled states are those states, which cannot be written in the above form in any basis. In a density matrix level, a state, \( \rho_{AB} \), is said to be entangled if

\[ \rho_{AB} \neq \sum_{i=1}^{d} p_i \rho_A^i \otimes \rho_B^i \]
where $p_i \geq 0, \forall i$, $\Sigma p_i = 1$, $\rho^i_A = |\psi^i_A\rangle\langle\psi^i_A|$ ($i = 1, \ldots d$) and similarly $\rho^i_{B} = |\psi^i_B\rangle\langle\psi^i_B|$ ($i = 1, \ldots d$) while separable state can be prepared by LOCC by Alia and Brato. An example of an entangled pure state is the single state, $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ where $|01\rangle$ and $|10\rangle$ represents eigenvectors of $\sigma_x$, with $\sigma_x$, $\alpha = x,y,z$ being the Pauli spin matrices. The singlet admixed with white noise represented by the identity operator, $I$, in the 4-dimensional Hilbert space, can be written as

$$\rho_w = p |\psi^-\rangle\langle\psi^-| + (1-p) \frac{I}{4}$$

(3)

It is shown to be entangled for certain value of mixing parameter $p$, specifically when $p > 1/3$. In 1989, R.F. Werner [17] possibly discussed about this state for the first time and show that there exists a range of $p$, namely $p \leq 1/\sqrt{2}$, in which $\rho_w$ is entangled but does not violate Bell inequality [18, 19] with two-settings.

It is important to note here that given a state, $\rho_{AB}$, it is not easy to prove whether the state is entangled or not. In case of bipartite pure states in smaller dimensions like for two spin-1/2 particles, a state can be shown to be entangled from the definition only. However, such a simple procedure may not work for arbitrary density matrix. Therefore, it is important to find entanglement detection methods. Over last 25 years, several mathematical as well as experimentally testable procedures have been found. Prominent entanglement detection methods [3, 4, 20] include partial transposition, majorization, covariance matrix criteria, and entanglement witness. Among these, we will discuss about the partial transposition criteria later.

Let us now move to a scenario where we want to describe entanglement properties of a state, $\rho_{12\ldots N}$, consisting of $N$ parties [21] situated in separate places. If a state can be written as

$$\rho_{12\ldots N} = \sum_i p_i \rho^{i}_1 \otimes \rho^{i}_2 \otimes \cdots \rho^{i}_N$$

(4)

it is called fully separable (useless). As one expects, $N$-party states can be classified in different ways. According to their entanglement content. A pure state is said to be $k$-separable [21] if it can be expressed as

$$|\psi_{12\ldots N}\rangle = |\psi_{1\ldots k-1}\rangle \otimes |\psi_{k+1\ldots N}\rangle$$

(5)

while a pure state is genuinely multiparty entangled if it is not product in any bipartition. Following this definition, in case of three parties, there exist three types of entangled states – fully separable, 2-separable or biseparable and genuinely multiparty entangled states. Two prominent examples of genuinely multipartite entangled states read as

$$|\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

(6)

which is known in the literature as Greenberger-Horne-Zeilinger state [22] and the W state [21,23,24], given by

$$|\psi_{\text{W}}\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |100\rangle + |101\rangle)$$

(7)

It was shown that genuine multiparty entangled states can be resource for several quantum information processing tasks like quantum secret sharing [25], multisite dense coding [26], measurement -based quantum computation [27].

1. Detecting entanglement: Partial Transposition Criteria

Among several entanglement detection methods proposed [28, 29], over the years partial transposition criteria turns out to be the efficient mathematical method to detect entanglement. For two qubits, it is necessary and sufficient to prove whether a state is entangled or not.

Let $C_A$ and $C_B$ be the complex Hilbert spaces of Alia and Brato and their corresponding dimensions be $N$ and $M$ respectively. If a bipartite density matrix can be represented as

$$\rho_{AB} = \sum_{i,j=1}^{N} \sum_{k,l=1}^{M} \alpha_{ij}^k |i\rangle_A \otimes |k\rangle_B$$

(8)

where $|i\rangle$ ($i = 1,2,\ldots,N$; $N \leq \dim C_A$) ($|k\rangle$ ($k = 1,2,\ldots,M$; $M \leq \dim C_B$)) is a set of real orthonormal vectors in $C_A$ ($C_B$).

The partial transposition of $\rho_{AB}$ with respect to subsystem $A$, denoted by $\rho_{T_A}^{AB}$, is defined as

$$\rho_{T_A}^{AB} = \sum_{i,j=1}^{N} \sum_{k,l=1}^{M} \alpha_{ij}^k |i\rangle_A \otimes |k\rangle_B$$

(9)

If $\rho_{T_A}^{AB}$ has any negative eigenvalue, we can then infer $\rho_{AB}$ to be entangled. For two-qubit entangled states, it can be shown that the partial transposed state has a single negative eigenvalue.

B. Quantifying Entanglement

Entanglement detection procedures can only give qualitative answers. Given a state, it is also essential to find the amount of resource that the state possess. For bipartite pure states, the von-Neumann entropy of local density matrices of a given state, $|\psi_{AB}\rangle$, can faithfully quantify the entanglement

$$E(|\psi_{AB}\rangle) = S(\rho_A)$$

(10)

Where $S(\sigma) = - tr(\sigma \log_2 \sigma)$ denotes the von Neumann entropy and $\rho_A = tr_B(|\psi_{AB}\rangle \langle \psi |)$ is the local density matrix of $|\psi_{AB}\rangle$. Examining convexity, we can now define entanglement for an arbitrary state, $\rho_{AB}$ as

$$E_F(\rho_{AB}) = \min \Sigma p_i (E(|\psi_i\rangle\langle \psi_i |))$$

(11)

with the minimization being performed over all possible pure state decomposition of $\rho_{AB}$ [30]. Since infinite number of such decompositions for $\rho_{AB}$ exist, it is, in general, not easy to compute. A compact form of $E_F$ for two qubits was found in Ref. [31].

Another important entanglement measure which can be computed in arbitrary dimension is logarithmic negativity based on partial transposition criteria [32]. Specifically, for an arbitrary density matrix, $\rho_{AB}$, it is defined as

$$L(\rho_{AB}) = \log_2 (2\Sigma |\lambda_i | + 1)$$

(12)

where $\lambda_i$ s are negative eigenvalues of partial transposed state of $\rho_{AB}$ with respect to $A$.

Although there are several entanglement measures introduced for two-qubit states, only a few multipartite computable entanglement measures are known in literature. For pure multipartite states, we introduce a measure which can quantify genuine multipartite entanglement content of the state [33, 34], known as generalized geometric measure (GGM). Let us define it for a three-party state,$|\psi_{ABC}\rangle$, which
can be easily be generalized to an arbitrary number of parties. It is defined as the minimum distance between a given state and non-genuinely multipartite entangled state. Mathematically,

$$G(\rho_{ABC}) = 1 - \max_{\sigma \in NG} \langle \sigma | \rho_{ABC} | \sigma \rangle^2, \quad (13)$$

where the maximization is performed over the set of all nongenuinely multipartite entangled states, denoted by NG. It can be shown that this measure can be computed for arbitrary number of parties because it can be simplified in terms of Schmidt coefficients of $|\psi_{ABC}\rangle$ in different bipartitions. Specifically,

$$G(\rho_{ABC}) = 1 - \max(\lambda_A^m, \lambda_B^m, \lambda_C^m)$$

where $\lambda_i^m$, $i = A, B, C$ represent the maximum eigenvalues of local density matrices of $\rho_i$, $i = A, B, C$ of $|\psi_{ABC}\rangle$. Notice that $i = A, B, C$ are the squares of the Schmidt coefficients of $|\psi_{ABC}\rangle$ in $A:B, B:C$ and $C:A$ bipartitions. For example, the GGM for the GHZ state, in Eq. (6), can be computed to be 0.5 since all the reduced local density matrices of the GHZ state is same due to symmetry and is 1/2, having equal eigenvalues, 0.5. It is important to note here that such a simplification does not hold if we extend it to a mixed state. However, it was recently found that for a certain classes of mixed states, GGM can be computed [35]. A compact form of GGM for arbitrary density matrices with arbitrary number of parties is still an open question.

**Generation of Resource States**

Let us discuss a physically realizable model in which resource states, i.e., entangled states can be created. To demonstrate that, let us consider a system in which $N$ spin 1/2 particles interact according to the Hamiltonian, given by [36]

$$H_{\text{Ising}} = -J \sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x + h \sigma_i^z, \quad (14)$$

where $J$ and $h$ represent respectively coupling constant and strength of the magnetic field, and periodic boundary condition is also considered, i.e., $\sigma_{N+1} = \sigma_1$. The above model is known as the Ising model which is one of the simplest model to undergo quantum phase transition at zero temperature and hence this model plays an important role in condensed matter physics. With the advancements of cold atoms, such a system can be prepared by using currently available technologies [15,37,38].

First note that the nearest-neighbor two-party density matrix of the zero-temperature state which is obtained by tracing out $N$-2 parties are all equal due to the periodic boundary condition. For example, to obtain $\rho_{23}$, as depicted in Fig. 1, one should trace out all the other sites except site 2 and 3 from a six-party state. We then can calculate its entanglement content (logarithmic negativity) in terms of system parameter, $\lambda = h/J$, as shown in Fig. 2 (see Ref. [39] for infinite spin chain and scaling of entanglement with $N$). One can check easily that next-nearest neighbor and other long-range entanglement is almost negligible or vanishing with the variation of $\lambda$.

**Figure 2:** Nearest neighbor entanglement of the zero temperature state of the Ising model (ordinate) vs. $\lambda = h/J$ (abscissa). The numerical simulation and the figure is produced by L.L. Ganesh Chandra.

![Figure 2](image2.png)

**Figure 3:** Nearest neighbor entanglement of the evolved state of the Ising model (ordinate) against $t$ (abscissa). The initial state is the canonical equilibrium state of the Ising Hamiltonian in Eq. (14) with $a = 0.5$ and $N = 6$. The numerical simulation and the figure is produced by L.L. Ganesh Chandra.

Let us now move to a situation where the system is initially prepared at the canonical equilibrium state with $\beta = 29.5$ of the Ising Hamiltonian. Then at $t > 0$, magnetic field, $h$ is turned off, i.e.,

$$h = a, \quad t \leq 0, \quad h = 0, \quad t > 0.$$
The system evolves according to the Ising Hamiltonian in Eq. (14) and can also generate nonvanishing nearest neighbor entanglement with the increase of time if one properly tunes the system parameter. Production of entanglement in this dynamical process with a = 0.5 is depicted in Fig. 3 with N = 6. In thermodynamic limit, i.e. for N→∞, one can find the behavior of bipartite entanglement for large time and can investigate the statistical mechanical properties like ergodicity in this model [40, 41]. It is important to note here that creating isolated system is almost impossible in laboratories – system inevitably interacts with the environment which can cause decay of entanglement. Currently, one of the active directions of research is to study the effects of environment on entanglement in these spin systems (see eg. [5, 42, 43, 45]).

Quantum Protocol with Entanglement at Resource

![Quantum Teleportation Protocol](image)

**Figure 4:** Quantum teleportation protocol. Steps involved in this protocol are schematically depicted where A is the sender having an unknown qubit A’ and B is the receiver.

As discussed before, entanglement is the necessary ingredient for several quantum communication protocols involving two or multiple parties. In this section, we illustrate one of them in which arbitrary quantum state is transferred from a sender to a receiver, known as quantum teleportation [46, 47].

Let us consider a scenario associating two parties, a sender called Alia (A) and a receiver called Brato (B) who are situated in different laboratories. Alia possesses another unknown qubit, A’, represented as |ψ⟩ = a|0⟩ + b|1⟩ where a and b are arbitrary complex number, satisfying the normalization condition, |a|^2 + |b|^2 = 1. Alia wants to send A’ to Brato [46] and she is not allowed to send it by using quantum channel. Therefore, we ask the following question: Is there a process by which a qubit can be transferred from Alia to Brato without sending it physically?

Let us suppose that Alia and Brato share an unentangled state. Since an arbitrary pure qubit can be any point in the surface of the Bloch sphere, Alia requires infinite amount of classical communication to encode such an arbitrary qubit, i.e. any complex pair.

Let us now consider a situation where Alia and Brato apriori share a singlet state, |ψ⟩, and she now wants to send A’ to Brato. Initially, Alia possesses two qubits – one part of a singlet and an unknown qubit. We describe the process (see Fig. 4 for pictorial illustration of the protocol) in the following:

**Step 1 (Entangling measurement):** Alia measures both her qubits in the Bell basis, \(|ψ⟩ = \frac{1}{\sqrt{2}} (|01⟩ - |10⟩)\) and \(|ϕ⟩ = \frac{1}{\sqrt{2}} (|00⟩ - |11⟩)\).

**Step 2 (2 bits of classical communication):** Alia communicates measurement results having 4 outcomes to Brato, i.e. she sends 2 bits of classical information to Brato.

**Step 3 (Decoding):** Depending on the measurement outcomes, Brato performs an unitary operation from the set, \{I, σ_x, σ_y, σ_z\}, on his qubit, as shown in Table below.

| Outcome | |ψ⟩ | |ψ⟩ | |ϕ⟩ | |ϕ⟩ |
|---------|---|---|---|---|---|
| Unitary | I | σ_x | σ_y | σ_z |

Remark: If Alia does not communicate about her measurement outcomes, the state at Brato’s side can be found to be an Identity operator of that space, having no information about the unknown qubit. It shows that there is no faster than light communication in this protocol. The above protocol.

The above protocol establishes that the singlet state is a resource in quantum teleportation. In laboratories, the shared state is in general mixed which have the fidelity with the singlet ranging from 80% to 99% depending on the physical systems used for implementations. The teleportation scheme with arbitrary mixed shared state has been addressed and the optimal teleportation fidelity is found [48, 49]. It has also been realized by using different physical systems like photons, ion traps [8–10, 47]. Point to point communication has very limited use. Therefore, considerable efforts have been put to extend such scheme in a multipartite setting [33].

**Conclusion**

Summarizing, we introduced one of the main resources available to build quantum technologies. A possible method to generate such resources was also shown. We then discussed a communication protocol which can only be successfully completed if the resource state is distributed among the parties.

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**References**

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11. The smallest unit in a classical computer which can take a binary value \(0, 1\), is called a bit (binary digit). Computation in a classical computer, smallest unit is a two-dimensional quantum system which is called quantum bit, in short qubit


19. J.S. Bell, Physics 1, 195 (1964)


44. We find the canonical equilibrium state, \(\frac{\exp(\beta H)}{Z}\), with \(Z = \text{tr} \exp(\beta H)\) being the partition function, Here \(\beta = \frac{1}{k_B T}\) where \(k_B\) is the Boltzmann constant and \(T\) is the temperature. Taking \(\beta \to \infty\) leads to the zero – temperature states


