# APPROXIMATING THE HARD CORE PARTITION FUNCTION WITH NEGATIVE ACTIVITIES

The objective of this document is to show that the analysis of correlation-decay algorithms for approximating the partition function of the hard core model on graphs of maximum degree at most $d+1$ extends easily to the case when all vertex activities are negative but strictly smaller than the Shearer threshold for graphs of degree at most $d+1$.

Recall that Weitz's method for approximating the partition function has three steps. The first step is to observe that standard self-reducibility arguments imply that in order to get an FPTAS for the partition function, it is enough to get an FPTAS for ratios of certain partition functions which correspond to the probability of a vertex *not* being in the independent set in the setting of positive vertex activities. While the probabilistic interpretation does not hold in the setting of negative edge activities, the self-reducibility arguments still go through using the positivity of the partition functions involved. The second step is based on Weitz's observation that the computation of these ratios on a given graph $G$ can be reduced to a similar computation on the self-avoiding walk tree of $G$: this part of the reduction also goes through without any changes.

The third step is to show that the computation can in fact be carried out on the self-avoiding walk tree truncated to have logarithmic depth, while incurring only an inverse polynomial loss in accuracy (carrying out the computation on the untruncated tree will give an exact answer, but will take exponential time). In previous analyses this step was carried out be showing that any errors introduced in the computation by an arbitrary initialization of the dynamic programming like recurrences for the ratios being computed decay by a constant factor *at each step* of the tree recurrence (so that after logarithmically many steps, we have only inverse polynomial error). The only part of the analysis that needs to be modified is this last: we need to show that that at each step of the recurrence, any errors in the "input" decay by a constant factor strictly smaller than 1, even when the activities are negative. However, the analysis in the case of negative activities actually turns out to be much simpler than the one in the well studied case of positive activities. This document describes this analysis.

## 1. The SAW tree recurrence

Weitz's tree recurrence holds without any changes even in the signed setting. In particular, given a tree rooted at a vertex $v$ with children $v_1, v_2, \ldots, v_d$, we have

$$R_v = f(R_1, R_2, \ldots, R_d) := \lambda \prod_{i=1}^{d} \frac{1}{1 + R_{v_i}}, \tag{1}$$

using Weitz's notation. In the notation of Scott and Sokal, $R_v$ is the quantity $\frac{1-p_v}{p_v}$ where $p_v := \frac{Z(\boldsymbol{\lambda}\mathbf{1}_{V-\{v\}})}{Z(\boldsymbol{\lambda})}$. Note that Theorem 2.10 of Scott and Sokal implies that all the ratios computed are negative in value, and indeed lie in the interval $(-1, 0]$, as long as all the activities satisfy Shearer's condition:

$$|\lambda_v| \leq \frac{d^d}{(d+1)^{d+1}}.$$

(Here, we assume that the graph is of degree at most $d+1$.)

We now note that at the leaves of the Weitz tree, we have $R_v = \lambda_v$ or $R_v = 0$ (if the vertex is fixed by Weitz's boundary conditions). We then have the following simple observation.

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**Observation 1.1.** *Let $G$ be a graph of degree at most $d+1$, and let $T(v, G)$ be the Weitz SAW tree of $G$ rooted at $v$. Let $u$ be any vertex in $T(v, G)$, and let $R_u$ be the value computed at the vertex $u$ in the tree computation. If there exists $c \in (0, 1)$ such that $0 \geq \lambda_v \geq -c \cdot \frac{d^d}{(d+1)^{d+1}}$ for all vertices $v$ in $G$ then $0 \geq R_u > -\frac{c}{d+1}$.*

*Proof.* We already observed that if $u$ is a leaf, we have $R_u = 0$ or $R_u = \lambda_u \geq -\frac{c}{d+1} \cdot \left(\frac{d}{d+1}\right)^d > -\frac{c}{d+1}$. We now proceed by induction on the height of $u$. Let $u_1, u_2, \ldots, u_k$ (for $k \leq d$) be the children of $u$. By induction, we can assume that $0 \geq R_{u_i} \geq -\frac{c}{d+1} > -\frac{1}{d+1}$. The recurrence in eq. (1) then immediately implies that $R_u \leq 0$ (since $\lambda_u \leq 0$). For the lower bound, we then have

$$\begin{aligned}
|R_u| &= |\lambda_u| \prod_{i=1}^{k} \frac{1}{1 + R_{u_i}} \\
&\leq \frac{c}{1+d} \left(\frac{d}{d+1}\right)^d \prod_{i=1}^{k} \frac{d+1}{d} \\
&\leq \frac{c}{1+d}, \text{ since } k \leq d.
\end{aligned}$$

□

We now look at the error in one step of the recurrence. Given two different vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of inputs to the one-step recurrence, we want to analyze the difference $|f(\boldsymbol{x}) - f(\boldsymbol{y})|$. (Note that we can assume that there are $d$ inputs since if there are less, then we can replaces the missing input by 0's without changing the output). We assume that the $\lambda_v$ satisfy the hypotheses in the above observation with $c$ as defined there, so that we can assert that the components of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are negative and at most $c/(d+1)$ in magnitude. We then use the mean value theorem to get

$$\begin{aligned}
|f(\boldsymbol{x}) - f(\boldsymbol{y})| &\leq \|\boldsymbol{x} - \boldsymbol{y}\|_\infty |f(\boldsymbol{z})| \sum_{i=1}^{d} \frac{1}{1 + z_i}, \text{ where } \boldsymbol{z} \text{ lies on the line segement joining } \boldsymbol{x} \text{ and } \boldsymbol{y}, \\
&\leq \|\boldsymbol{x} - \boldsymbol{y}\|_\infty |f(\boldsymbol{z})| (d+1), \text{ using } z_i \geq -\frac{c}{d+1} > -\frac{1}{d+1} \\
&\leq c \cdot \|\boldsymbol{x} - \boldsymbol{y}\|_\infty,
\end{aligned}$$

where in the last line we use the bounds $z_i > -\frac{1}{d+1}$ and $|\lambda_v| \leq c \cdot \frac{d^d}{(d+1)^{d+1}}$ to conclude that $|f(z)| \leq \frac{c}{d+1}$, as in the proof of the above observation.

To show that this yields an FPTAS whenever $c < 1$, we note that this gives an additive FPTAS for $R_v$. Define $p_v$ to the positive quantity given by $\frac{Z(\boldsymbol{\lambda} \mathbf{1}_{V-\{v\}})}{Z}$ (the positivity follows from Theorem 2.10 of Scott and Sokal). Then, we have $R_v = \frac{1}{p_v} - 1$. Using the absolute bounds on $R_v$, we note that an additive FPTAS for $R_v$ provides a multiplicative FPTAS for $p_v$, which in turn provides an FPTAS for $Z$ using the usual self-reducibility procedure.