A simplified proof of a Lee-Yang type theorem

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In this short note, we give a simple proof of a Lee-Yang type theorem which appeared in [SS14]. Given an undirected graph $G = (V, E)$, we denote the partition function of the (ferromagnetic) Ising model as

$$Z(G, \beta, \boldsymbol{z}) := \sum_{\sigma: V \to \{+,-\}} \beta^{d(\sigma)} \prod_{v:\sigma(v)=+} z_v, \qquad (\boldsymbol{z} = (z_v)_{v \in V})$$

where $d(\sigma)$ is the number of edges $e = \{i, j\}$ such that $\sigma(i) \neq \sigma(j)$, and $0 < \beta < 1$ is the *edge activity*. The arguments z_i of the partition function are called *vertex activities* or *fugacities*. We then define the operator $\mathcal{D}_G$

$$\mathcal{D}_G = \sum_{v \in V} z_v \frac{\partial}{\partial z_v},$$

which derives its importance from the fact that the *mean magnetization* $M(G, \beta, \boldsymbol{z})$ of the Ising model on G for a given setting of the edge activity and the fugacities can be written as

$$M(G, \beta, \boldsymbol{z}) = \frac{\mathcal{D}_G Z(G, \beta, \boldsymbol{z})}{Z(G, \beta, \boldsymbol{z})}.$$

The theorem whose proof in this note we simplify is the following:

Theorem 1 ([SS14]). *Let $G = (V, E)$ be a connected undirected graph on n vertices, and assume $0 < \beta < 1$. Then $\mathcal{D}_G Z(G, \beta, \boldsymbol{z}) \neq 0$ if for all $v \in V$, z_v is a complex number with absolute value one.*

In [SS14], the theorem was proved using a sequence of Asano-type contractions [Asa70], a technique which originated in Asano's proof of the Lee-Yang theorem [LY52]. The proof we present here completely eschews the Asano contraction in favor of a simpler analytic argument. In our proof we need the following version of the Lee-Yang theorem:

Theorem 2 ([LY52, Asa70]). *Let $G = (V, E)$ be a connected undirected graph on n vertices, and suppose $0 < \beta < 1$. Then $Z(G, \beta, \boldsymbol{z}) \neq 0$ if $|z_v| \geq 1$ for all $v \in V$ and in addition $|z|_u > 1$ for some $u \in V$. By symmetry, the conclusion also holds when $|z_v| \leq 1$ for all $v \in V$ and in addition $|z|_u < 1$ for some $u \in V$.*

Observe that given any vertex $u \in V$, we can decompose the partition function as

$$Z(G, \beta, \boldsymbol{z}) = A\, z_u + B \tag{1}$$

$$A = A(\boldsymbol{z}) = \beta^{\deg(u)} Z(G - \{u\}, \beta, \boldsymbol{z}') \qquad z'_w = \begin{cases} z_w & \text{when } w \not\sim u \text{ in } G \\ z_w/\beta & \text{when } w \sim u \text{ in } G \end{cases} \tag{2}$$

$$B = B(\boldsymbol{z}) = Z(G - \{u\}, \beta, \boldsymbol{z}'') \qquad z''_w = \begin{cases} z_w & \text{when } w \not\sim u \text{ in } G \\ z_w \beta & \text{when } w \sim u \text{ in } G \end{cases}$$

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Neither $\boldsymbol{z}'$ nor $\boldsymbol{z}''$ contains z_u and $G - \{u\}$ denotes the graph that we obtain from G by leaving out node u. The Lee-Yang theorem has the following simple consequence, which was also used in [SS14].

Lemma 3. *If G is connected, $0 < \beta < 1$, and all vertex activities have absolute value 1, then A of eq. (2) is not zero.*

Proof. Since $\beta \neq 0$, it is sufficient to prove that $Z(G - \{u\}, \beta, \boldsymbol{z}') \neq 0$. We observe that the latter is a product of the partition functions of the connected components of $G - \{u\}$, and furthermore, any neighbor w of u in G in each such component has a vertex activity $z'_w = z_w/\beta$ with $|z_w/\beta| = |z_w|/\beta = 1/\beta > 1$. Due to G being connected, we find such a neighbor w of u in all components of $G - \{u\}$. We apply Theorem 2 to each connected component of $G - \{u\}$ separately to show that none of the factors is zero. □

Proof of Theorem 1. Let G and β be as in the hypotheses of the theorem. Suppose now that there exists a point $\boldsymbol{z^0}$ such that $\left|z_v^0\right| = 1$ for all v, and $\mathcal{D}_G Z(G, \beta, \boldsymbol{z^0}) = 0$. We will show that this leads to a contradiction. For our subsequent argument it will be helpful to define the univariate polynomial

$$f(t) := Z_G(G, \beta, t\boldsymbol{z^0}) \qquad \text{where} \qquad t\boldsymbol{z^0} = \left(tz_v^0\right)_{v \in V}$$

Lemma 4. $Z(G, \beta, \boldsymbol{z^0}) = 0$

Proof. A comparison of the individual terms gives that $f'(1) = \mathcal{D}_G Z(G, \beta, \boldsymbol{z^0})$, which is zero by our assumption. From the Lee-Yang theorem we obtain that $f(t) \neq 0$ when $|t| \neq 1$, so all zeros of f must lie on the unit circle. This together with the Gauss-Lucas lemma implies that the derivative of f cannot disappear on a point of the unit circle unless f disappears at the same point. Thus, since $f'(1) = 0$, we get that $Z(G, \beta, \boldsymbol{z^0}) = f(1) = 0$. □

We have that $f(1 - \epsilon) = f(1) - \epsilon f'(1) \pm O(\epsilon^2) = \pm O(\epsilon^2)$, since the first two terms are zero. Let $\boldsymbol{e_u}$ be the vertex activity (fugacity) vector with all zero vertex activities except at vertex u that has activity 1. The key to the proof is to consider the linear perturbation

$$Z(G, \beta, (1 - \epsilon)\boldsymbol{z^0} + \tau \boldsymbol{e_u}) \tag{3}$$

We show that (3) disappears for some $\tau \in \mathbb{C}$, $|\tau| < \epsilon$, in contradiction with the Lee-Yang theorem, since under this assumption all components of $(1 - \epsilon)\boldsymbol{z^0} + \tau \boldsymbol{e_u}$ have absolute value less than one. By (1):

$$Z(G, \beta, (1 - \epsilon)\boldsymbol{z^0} + \tau \boldsymbol{e_u}) = Z(G, \beta, (1 - \epsilon)\boldsymbol{z^0}) + A((1 - \epsilon)\boldsymbol{z^0})\tau = \mu_\epsilon + A(\boldsymbol{z^0})\tau + \nu_\epsilon \tau$$

Here $\mu_\epsilon = f(1 - \epsilon) = \pm O(\epsilon^2)$, and $\nu_\epsilon = A((1 - \epsilon)\boldsymbol{z^0}) - A(\boldsymbol{z^0}) = \pm O(\epsilon)$ by the analyticity of the function A. Recall that $A(\boldsymbol{z^0}) \neq 0$ by Lemma 3. Then expression (3) disappears at $\tau = -\frac{\mu_\epsilon}{A(\boldsymbol{z^0}) + \nu_\epsilon}$, and also $|\tau| < \epsilon$ if ϵ is sufficiently small. □

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