Toolkit for TCS — **Monsoon 2018**

HW 3

Out: November 1, 2018 — **Due:** November 15

(1) Please take time to write clear and concise solutions. You are STRONGLY *encouraged to submit LaTeXed solutions by email. (2) Collaboration is OK, but please write your answers yourself, and include in your answers the names of* EVERYONE *you collaborated with and* ALL *references other than class notes you consulted.*

1. (2 points) Give an example of an *asymmetric* matrix A such that for all real vectors v of appropriate shape, $v^T Av \geq 0$.

2. (2 points) If A is symmetric, show that

$$\lambda_{\min}(A) = \max \{\mu \mid A - \mu I \succeq 0\},$$

and

$$\lambda_{\max}(A) = \min \{\mu \mid \mu I - A \succeq 0\}.$$

3. (2 points) Let A be the adjacency matrix of an undirected graph G, and let v be its top eigenvector with eigenvalue κ. Note that $\kappa \geq 0$ and v can be so chosen as to have non-negative entries. Let us assume further that we write v so that its entries are arranged in descending order (so that $v_1 \leq v_2 \leq \cdots \leq v_n$). Note that this induces an ordering on the vertices of G.

   Consider now the following colouring procedure, which is based on the above order. We start with an empty list L of colours. We then process the vertices $u_1, u_2, \ldots, u_n$ in order, and for any given i, construct the set S of the colours assigned to neighbors u_j of u_i with $j < i$. If $S = L$, then we create a new color c, set $L = L \cup \{c\}$ and assign the colour c to u_i. Otherwise, if $L \setminus S$ is non-empty, we choose a color from $L \setminus S$ (according to some pre-defined choice rule) and assign it to u_i. Note that this procedure produces a proper coloring of the graph.

   How large can L can be at the end of the algorithm? Show that your bound is tight by giving an appropriate example.

For the following three problems, let $G = (V, E)$ be an undirected graph and let $L = D - A$ be its Laplacian. The largest eigenvalue of the normalized Laplacian, denoted by ν_n satisfies

$$\nu_n = \max_{x \neq 0} \frac{x^T Lx}{x^T Dx}.$$

4. (4 points) **[bipartite $\Leftrightarrow$ $\nu_n = 2$]**

   Prove that $\nu_n \leq 2$. Furthermore, prove that equality holds iff the graph G is bipartite.

   Hint: $(x_u - x_v)^2 = 2(x_u^2 + x_v^2) - (x_u + x_v)^2$.

5. (4 points) **[almost bipartite $\Rightarrow$ ν_n almost 2]**

   Suppose the MAXCUT in G has normalized cost at least $1 - \varepsilon$. That is, there exists a cut $(S, V \setminus S)$ such $\partial S \geq (1 - \varepsilon)|E|$. Prove that there is a non-zero vector $x \in \mathbb{R}^V$ such that

$$x^T (D + A)x \leq 2\varepsilon \cdot x^T Dx.$$

   Hence, conclude that $\nu_n \geq 2(1 - \varepsilon)$.

6. (14 points) **[ν_n almost 2 $\Rightarrow$ almost bipartite]**

   In this problem, we will prove the following theorem.

   Theorem. *Let $\nu_n \geq 2(1 - \varepsilon)$ or equivalently there exists a non-zero vector $x \in \mathbb{R}^V$ such that $x^T(D + A)x \leq 2\varepsilon \cdot x^T Dx$. Then there exists non-zero vector $y \in \{-1, 0, 1\}^V$ such that*

$$\frac{\sum_{\{u,v\} \in E} |y_u + y_v|}{\sum_{u \in V} d_u |y_u|} \leq \sqrt{8\epsilon}.$$

To this end, we define the following randomized process that constructs a random non-zero vector $Y \in \{-1, 0, 1\}^V$ given a non-zero vector $x \in \mathbb{R}^V$ satisfying $x^T(D + A)x \leq 2\varepsilon \cdot x^T Dx$. Since this latter condition is scale-invariant, we may assume wlog. that $\max_u |x_u| = 1$ and let $u_* \in V$ such that $|x_{u_*}| = 1$.

- Pick a value t uniformly in $[0, 1]$.
- Define $Y \in \{-1, 0, 1\}^V$ as follows:

$$Y_u = \begin{cases} -1 & \text{if } x_u < -\sqrt{t}, \\ 1 & \text{if } x_u > \sqrt{t}, \\ 0 & \text{otherwise, i.e., } |x_u| \leq \sqrt{t}. \end{cases}$$

(a) (2 points) Prove that $\mathbf{P}[\exists u \in V, Y_u \neq 0] = 1$.

(b) (3 points) Prove that $\mathbf{E}[|Y_u|] = x_u^2$ and $\mathbf{E}[|Y_u + Y_v|] \leq |x_u + x_v| \cdot (|x_u| + |x_v|)$.

(c) (6 points) Prove that $\mathbf{E}\left[\sum_{\{u,v\} \in E} |Y_u + Y_v|\right] \leq \sqrt{8\epsilon}\,\mathbf{E}\left[\sum_u d_u |Y_u|\right]$.

Hint: Cauchy-Schwarz Inequality.

(d) (3 points) Hence, conclude that there exists a non-zero vector $y \in \{-1, 0, 1\}^V$ such that $\sum_{\{u,v\} \in E} |y_u + y_v| \leq \sqrt{8\epsilon}. \cdot \sum_{u \in V} d_u |y_u|$.

Discussion. *It is known that G is connected if $\nu_2 \neq 0$. Or equivalently, $\phi(G) \neq 0$ iff $\nu_2 \neq 0$. Cheeger's inequalities give a "quantitative strengthening" of this statement by showing that*

$$\sqrt{2\nu_2} \geq \phi(G) \geq \nu_2/2.$$

The above three problems are similar in spirit but work with ν_n and "bipartiteness" instead of ν_2 and "connectedness".

Define the bipartiteness ratio number *of a graph G to be*

$$\beta(G) := \min_{y \in \{-1,0,1\}^V} \frac{\sum_{\{u,v\} \in E} |y_u + y_v|}{2d \sum_{u \in V} |y_u|},$$

which is equivalent to

$$\beta(G) = \min_{S \subseteq V, (L,R) \text{ partition of } S} \frac{2\partial(L, L) + 2\partial(R, R) + \partial(S, V \setminus S)}{d|S|},$$

Observe that $\beta(G) = 0$ iff G is bipartite. Problem 4 shows that $\beta(G) = 0$ iff $\nu_n = 2$. Problems 5–6 are a quantitative strengthening of this claim as they demonstrate that

$$\sqrt{2(2 - \nu_n)} \geq \beta(G) \geq \frac{1}{2} \cdot (2 - \nu_n).$$

These 2 problems are due to a result by Luca Trevisan.