Zeros of polynomials and graphs

– So far we saw spectral techniques based primarily on the Courant-Fisher variational characterization of eigenvalues. Today we look at ~~the~~ a different characterization of eigenvalues in terms of zeros of the characteristic polynomial. In particular, eigenvalues of any matrix A are roots of

$$\chi_A(\lambda) = \det(A - \lambda I) = 0 \text{ in } \lambda.$$

~~The~~ When the matrix A is symmetric, all roots of χ_A are real.

This view of the spectrum, though classical, has ~~been~~ led to some striking results in theoretical computer science in the past decade.

We look at one such application: Sparsification [Batson, Spielman, Srivastava '08]

Sparsification :- Let L_G be the Laplacian of an undirected graph. Another weighted graph H is ~~st~~ said to be a κ-spectral sparsifier of G if

$$x^T L_H x \leq x^T L_G x \leq K \cdot x^T L_H x .$$

This is a strong notion :- Consider any cut $(=(S, \bar{S})$ in G. Then it is easy to see that ~~the wts of~~

$$\text{wt. of } C \text{ in } G = 1_S^T L_G 1_S$$
$$\text{wt. of } C \text{ in } H = 1_S^T L_H 1_S$$

So that

$$1 \leq \frac{wt_G(C)}{wt_H(C)} \leq K .$$

["Cut sparsification"]

We will prove the following simpler result. Let $u_1 \ldots \ldots u_m \in \mathbb{R}^n$ be a set of vectors s.t.

$$\sum_{i=1}^{m} v_i v_i^T = I .$$

Then $\exists$ ~~a~~ s_i, $1 \le i \le m$, s.t. $|\{ \cancel{i}\; s_i \neq 0\}| \le \lceil (n-1)d \rceil$ s.t.

$$I \le \sum_{i=1}^{m} s_i v_i v_i^T \le \cancel{\left(1+\frac{c}{d}\right) I_6} \left(1+\frac{8}{d}\right) I$$

~~(~~ RHS can be improved to $= 1 + \frac{2}{\sqrt{d}}$ ..

The algorithm would be of the following form: ~~For (n-1)d~~ Starting with $M = 0$, it will add $\lceil \cancel{(n-1)d}\, nd \rceil$ vectors ~~with~~ chosen from the v_i, with s_i chosen appropriately, to ~~the~~ M.

We thus need to analyze how the spectrum of M changes with each "rank 1-update" of the form

$$M \mapsto M + s_i v_i v_i^T.$$

Rank 1 updates

Inversion:- The Sherman-Morrisson formula.

Consider the inverse B of $A = I + uv^T$.

Note that if $w \perp v$, then $Aw = w$, while

$Av = v + (v^T v)\, u$.

Thus B is defined uniquely by

$$Bw = w \quad \text{for } w \perp v.$$

$$\cancel{Bv + B(v^T v)u =}$$

$$Bv + (v^T v)\, Bu = v.$$

Let $u = w + \beta v$ where $w \perp v$.

Then $\beta = \dfrac{u^T v}{v^T w}$, $w = u - \dfrac{(u^T v)}{(v^T v)} v$.

$$Bu = u - \left(\frac{u^T v}{v^T v}\right) v + \left(\frac{u^T v}{v^T v}\right) Bv \;..$$

Hence $Bv + (v^T v)u - (u^T v)v + (u^T v)Bv = v$

$$(1+u^Tv)\,Bv = (1+u^Tv)\,v - (v^Tv)\,u.$$

and $$Bv = v - \left(\frac{v^Tv}{1+u^Tv}\right)u.$$

[Note that this is essentially a two dimensional problem: completely located in the span of v and u.]

Thus we have

$$Bv = v - \frac{v^Tv}{1+u^Tv}\,u$$

$$Bw = w \quad \forall\ w \perp v.$$

So, $B = A^{-1} = I - \dfrac{\cancel{v^Tv}\,uv^T}{1+u^Tv}$ [Q: What happens when $1+u^Tv=0$?]

So,

$$\boxed{(I+uv^T)^{-1} = I - \frac{uv^T}{1+u^Tv}.}$$

<u>Verification</u>:- $\left(I - \dfrac{uv^T}{1+u^Tv}\right)(I+uv^T)$

$$= I + uv^T - \frac{uv^T}{1+u^Tv} - \frac{u^Tv\,uv^T}{1+u^Tv} = I.$$

Consider now the general case:

$$(A+uv^T)^{-1} = \left[A(I+A^{-1}uv^T)\right]^{-1}$$ (assuming A is invertible and so is $I+A^{-1}uv^T$)

$$= (I + A^{-1}uv^T)^{-1} A^{-1}$$

$$= \left(I - \frac{A^{-1}uv^T}{1+u^TA^{-T}v}\right)A^{-1}$$

$$= A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}.$$

Rank one update: the determinant.

Again let us consider

$\det(I + uv^T)$.

Consider writing this matrix in ~~the ba~~ an orthonormal basis $\hat{v}_1, \hat{v}_2, \ldots \hat{v}_n$ where $\hat{v}_1 \parallel v$. Then, $I + uv^T$ is an upper triangular matrix, with ~~non~~ all diagonal entries except the first equal to 1. The first diagonal entry is equal to $1 + v^Tu$.

Then $\det(I + uv^T) = 1 + v^T u$.

Now if A is invertible then

$$\det(A + uv^T) = \det(A(I + A^{-1}uv^T))$$
$$= \det(A)\,(1 + v^T A^{-1} u).$$

An immediate corollary of this is an equation for the e.v.'s of a rk. 1 update.

Let A be a symmetric matrix with eigenvalues $(\lambda_i)_{i=1}^n$ with corresponding unit eigenvectors $(u_i)_{i=1}^n$. ~~Then~~, Consider a ~~det~~ rank-1 update of the form $A + svv^T$.

Then

$$\det(A + svv^T - \lambda I) = \det(A - \lambda I)\left(1 + s v^T \cancel{A^{-1}} (A-\lambda I)^{-1} v\right)$$

$$= \det(A - \lambda I)\left(1 + s v^T \sum_{i=1}^n \frac{1}{\lambda_i - \lambda} u_i u_i^T v\right)$$

$$= \det(A - \lambda I)\left(1 + \sum_{i=1}^n \frac{s|v^T u_i|^2}{\lambda_i - \lambda}\right)$$

So, $$\chi_{A + svv^T}(\lambda) = \chi_A(\lambda)\left(1 + \sum_{i=1}^n \frac{s\,|v^T u_i|^2}{\lambda_i - \lambda}\right)$$