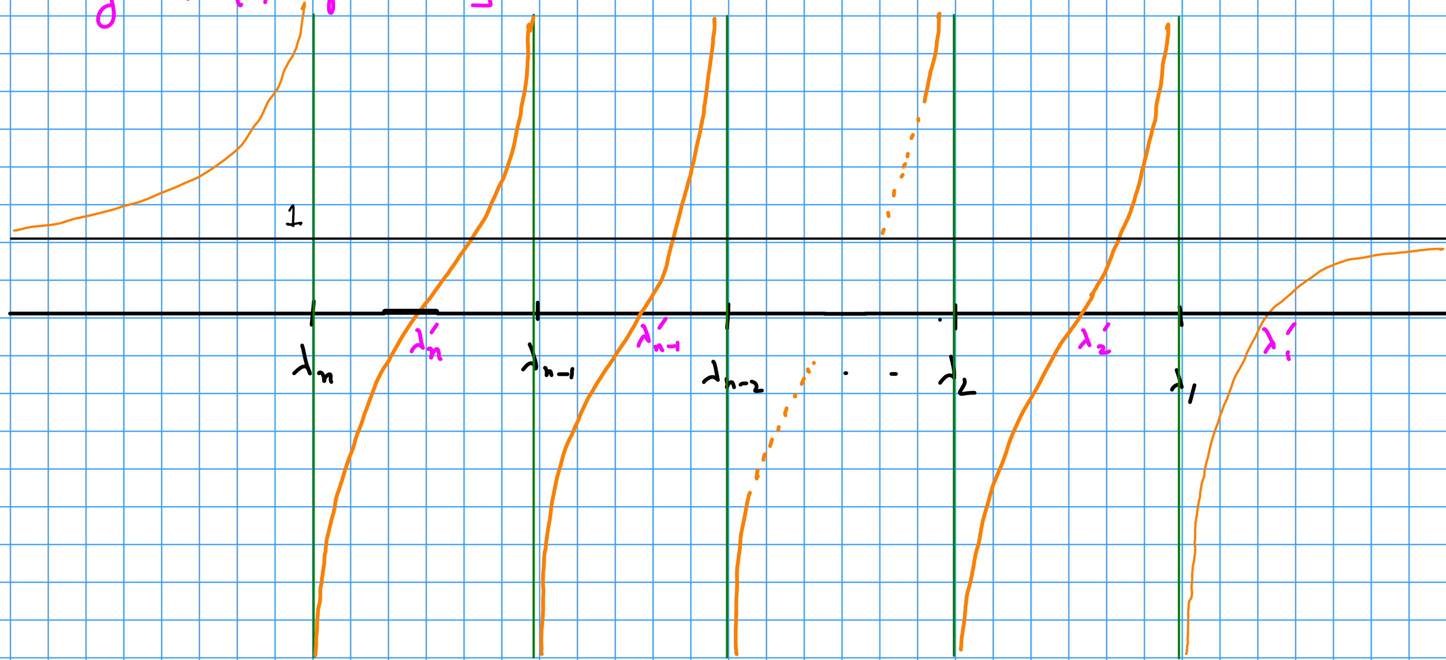
Consider $f_{v,A}(\lambda) = 1 + \sum_{i=1}^{n} \frac{|v^T u_i|^2}{\lambda_i - \lambda}$ where $A = \sum_{i=1}^{n} \lambda_i u_i u_i^T$, u_i orthonormal, $\lambda_i \in \mathbb{R}$.

[Assuming $v^T u_i \neq 0$ for $\forall 1 \leq i \leq n$]



<u>Outline of proof</u>:- Consider two "barrier functions"

$$\Phi^u(A) := Tr\left((uI - A)^{-1}\right) = \sum_{i=1}^{n} \frac{1}{u - \lambda_i}$$

$$\phi_\ell(A) := Tr\left((A - \ell I)^{-1}\right) = \sum_{i=1}^{n} \frac{1}{\lambda_i - \ell}.$$

Note that as long as $\lambda_1(A) < u$, $\Phi^u(A) \geq 0$ and is finite and as long as $\lambda_n(A) > \ell$, $\phi_\ell(A) \geq 0$ and is finite.

Further, as $u \downarrow \lambda_1(A)$ (respectively, $\ell \uparrow \lambda_n(A)$),
$\Phi^u(A) \uparrow \infty$ $(\phi_\ell(A) \uparrow \infty)$.

The general strategy is now as follows:- At the beginning of step q, we have

$$\Phi^{u_q}(A^{(q)}) \leq \varepsilon_u \quad \text{where } u_q = u_0 + (q-1)\delta_u$$

$$\phi_\ell^{\ell_q}(A^{(q)}) \leq \varepsilon_\ell \quad \text{where } \ell_q = \ell_0 + (q-1)\delta_\ell$$

We then try to obtain $A^{(q+1)} = A^{(q)} + t\, v v^T$ in such a way that

$$\Phi^{u+q\delta_u}\left(A^{(q+1)}\right) \leq \varepsilon_u$$

$$\Phi^{l+q\delta_u}\left(A^{(q+1)}\right) \leq \varepsilon_l$$

and $\quad 0 \leq l_0 + q\,\delta_u \leq \lambda_{min}\left(A^{(q+1)}\right) \leq \lambda_{max}\left(A^{(q+1)}\right) \leq u_0 + q\,\delta_u$

Here $\delta_u, \delta_l, l_0, u_0$ [and $\varepsilon_u, \varepsilon_l$] are to be chosen such that

$$\frac{u_0 + nd\,\delta_u}{l_0 + nd\,\delta_u} \quad \text{is at most a small constant.} \quad \left[\approx 1 - \frac{2}{\sqrt{d}}\right]$$

To implement this strategy, we first need to understand how the potential changes with rank one updates.

We have [UPPER BARRIER]

$$\Phi^u(A + tvv^T) = \mathrm{Tr}\left((uI - A - tvv^T)^{-1}\right)$$

$$= \mathrm{Tr}\left((uI - A)^{-1} + \frac{t\,B^{-1}vv^TB^{-1}}{1 - tv^TB^{-1}v}\right) \quad \text{where } B = uI - A$$

assume $B \succcurlyeq 0$ [Why?]

$$= \Phi^u(A) + \frac{t\,\mathrm{Tr}(B^{-1}vv^TB^{-1})}{1 - tv^TB^{-1}v} = \Phi^u(A) + \frac{v^TB^{-2}v}{1/t - v^TB^{-1}v}.$$

Thus,

$$\Phi^{u+\delta}(A + t\,vv^T) = \Phi^u(A) - \left(\Phi^u(A) - \Phi^{u+\delta}(A)\right) + \frac{v^T((u+\delta)I - A)^{-2}v}{1/t - v^T((u+\delta)I - A)^{-1}v}$$

In particular, if $t > 0$ and

$$\frac{1}{t} \geq \underbrace{\frac{v^T((u+\delta)I - A)^{-2}v}{\Phi^u(A) - \Phi^{u+\delta}(A)} + v^T((u+\delta)I - A)^{-1}v}_{U_{A,u+\delta}(v),\ \text{linear in } vv^T.}$$

[Why?]

then $\Phi^{u+\delta}(A+tvv^T) \leq \Phi^u(A)$ — ①

for **all** small enough positive t

Now for the LOWER BARRIER

$$\phi_\ell(A+tvv^T) = \mathrm{Tr}\left((A+tvv^T-\ell I)^{-1}\right)$$
$$= \mathrm{Tr}\left((A-\ell I)^{-1} - \frac{t\,B^{-1}vv^TB^{-1}}{1+tv^TB^{-1}v}\right) \text{ where } B = A-\ell I.$$
$$= \phi_\ell(A) - \frac{v^TB^{-2}v}{1/t + v^TB^{-1}v}.$$

Thus,

$$\phi_{\ell+\delta}(A+tvv^t) = \phi_\ell(A) + \left(\phi_{\ell+\delta}(A) - \phi_\ell(A)\right) - \frac{v^T(A-(\ell+\delta)I)^{-2}v}{1/t + v^T(A-(\ell+\delta)I)^{-1}v}$$

Provided δ is small enough, we have

$$A \succcurlyeq (\delta+\ell)I \quad [\text{provided } A \succ \ell I].$$

Under this condition if

$$0 < \frac{1}{t} < \underbrace{\frac{v^T(A-(\ell+\delta)I)^{-2}v}{\phi_{\ell+\delta}(A) - \phi_\ell(A)} - v^T(A-(\ell+\delta)I)^{-1}v}_{L_{A,\ell+\delta}(v)}.$$

then $\phi_{\ell+\delta}(A+tvv^T) \leq \phi_\ell(A)$

Thus, provided t is not too small, we have

$$\phi_{\ell+\delta}(A+tvv^T) \leq \phi_\ell(A). \quad \text{— ②}$$

Note: ① and ② are competing requirements

Goal :- Find $\delta_\ell, \delta_u, \varepsilon_\ell, \varepsilon_u, u_0, l_0$ s.t. at each step one can find v and t for which both ① and ② hold.

Formal statements of observations so far

If $\lambda_{max}(A) < u$, $\underbrace{\lambda_{min}(A) > l \text{ and } \phi_\ell(A) \leq 1/\delta_\ell}_{\text{Together imply } \lambda_{min}(A) > l+\delta_\ell}$,

and t, v are such that

$$0 \leq U_{A, u+\delta_u}(v) \leq \frac{1}{t} \leq L_{A, l+\delta_L}(v)$$

then,

$$\overline{\Phi}^{u+\delta_u}(A+tvv^T) \leq \overline{\Phi}^{u}(A) \text{ and } \lambda_{max}(A+tvv^T) \leq u+\delta_u$$

$$\Phi_{l+\delta_\ell}(A+tvv^T) \leq \phi_\ell(A) \text{ and } \lambda_{min}(A+tvv^T) \geq l+\delta_\ell.$$

In particular if the above conditions hold for $u = u_i$ and $l = l_i$ and we can find a corresponding t_i and v_i for $1 \leq i \leq nd$ then,

$$\frac{\lambda_{max}(A^{(nd)})}{\lambda_{min}(A^{(nd)})} \leq \frac{u_{nd}}{l_{nd}} = \frac{u_0 + nd\,\delta_u}{l_0 + nd\,\delta_\ell}$$

To show that such t and v exist we will show that under appropriate conditions on $\delta_\ell, \delta_u, \varepsilon_\ell, \varepsilon_u$, and u, l, A s.t.

$$\lambda_{max}(A) < u, \quad \lambda_{min}(A) > l,$$

$$\overline{\Phi}^{u}(A) \leq \varepsilon_u, \quad \phi_\ell(A) \leq \varepsilon_\ell,$$

and any collection of vectors v s.t.

$$\sum_{j=1}^{m} v_j v_j^T = I,$$

it holds that:

(*) $$\sum_j L_{A, l+\delta_\ell}(v_j) \geq \sum_j U_{A, u+\delta_u}(v_j) \geq 0$$

From this, it follows that there exists v_j, t s.t.

$$L_{A,\, l+\delta_l}(v_j) \geq \frac{1}{t} \geq U_{A,u+\delta_u}(v_j) \quad \text{[WHY?]}$$

For ⊛, we compute

$$\sum_j U_{A,u+\delta_u}(v_j) = \frac{\operatorname{Tr}\left((u+\delta_u I - A)^{-2}\left(\sum_j v_j v_j^T\right)\right)}{\Phi^u(A) - \Phi^{u+\delta_u}(A)} + \operatorname{Tr}\left((u+\delta_u I - A)^{-1}\left(\sum_j v_j v_j^T\right)\right)$$

$$= \frac{\operatorname{Tr}\left((u+\delta_u I - A)^{-2}\right)}{\Phi^u(A) - \Phi^{u+\delta_u}(A)} + \operatorname{Tr}\left((u+\delta_u I - A)^{-1}\right)$$

$$= \frac{\sum_i (u+\delta_u - \lambda_i(A))^{-2}}{\sum_i (u-\lambda_i(A))^{-1} - \sum_i (u+\delta_u - \lambda_i(A))^{-1}} + \sum (u+\delta_u - \lambda_i(A))^{-1}$$

$$= \frac{\sum_i (u+\delta_u - \lambda_i(A))^{-2}}{\delta_u \sum_i (u-\lambda_i(A))^{-1}(u+\delta_u - \lambda_i(A))^{-1}} + \sum (u+\delta_u - \lambda_i(A))^{-1}$$

$$\leq \frac{1}{\delta_u} + \Phi^{u+\delta_u}(A) \leq \frac{1}{\delta_u} + \varepsilon_u .$$

$$\left[\because \ \Phi^{u+\delta_u}(A) \leq \Phi^u(A) \leq \varepsilon_u\right]$$

Similarly

$$\sum_j L_{A,\, l+\delta_l}(v_j) = \frac{\sum_i (\lambda_i(A) - l - \delta_l)^{-2}}{\sum_i (\lambda_i(A) - l - \delta_l)^{-1} - \sum_i (\lambda_i(A) - l)^{-1}} - \sum_i (\lambda_i(A) - l - \delta_l)^{-1}$$

$$= \frac{\sum_i (\lambda_i(A) - l - \delta_l)^{-2}}{\delta_l \sum_i (\lambda_i(A) - l - \delta_l)^{-1}(\lambda_i(A) - l)^{-1}} - \sum (\lambda_i(A) - l - \delta_l)^{-1}$$

$$= \frac{1}{\delta_\ell} + \left(\frac{\sum_i (\lambda_i(A)-\ell-\delta_\ell)^{-2} (\lambda_i(A)-\ell)^{-1}}{\sum_i (\lambda_i(A)-\ell-\delta_\ell)^{-1} (\lambda_i(A)-\ell)^{-1}} \right) - \sum (\lambda_i(A)-\ell-\delta_\ell)^{-1}$$

$$= \frac{1}{\delta_\ell} - \sum (\lambda_i(A)-\ell)^{-1} + \left(\frac{\sum_i (\lambda_i(A)-\ell)^{-1} (\lambda_i(A)-\ell-\delta_\ell)^{-2}}{\sum_i (\lambda_i(A)-\ell)^{-1} (\lambda_i(A)-\ell-\delta_\ell)^{-1}} \right) - \delta_\ell \sum_i (\lambda_i(A)-\ell)^{-1} (\lambda_i(A)-\ell-\delta_\ell)^{-1}.$$

$$\geq \frac{1}{\delta_\ell} - \sum_i (\lambda_i(A)-\ell)^{-1} \geq \frac{1}{\delta_\ell} - \varepsilon_\ell \quad \text{when } \delta_\ell \sum (\lambda_i(A)-\ell)^{-1} \leq 1,$$

since then, by Hölder's inequality:

$$\delta_\ell \left(\sum_i (\lambda_i(A)-\ell)^{-1} (\lambda_i(A)-\ell-\delta_\ell)^{-1} \right)^2 \leq \delta_\ell \left(\sum_i (\lambda_i(A)-\ell)^{-1} \right) \left(\sum_i (\lambda_i(A)-\ell)^{-1} (\lambda_i(A)-\ell-\delta_\ell)^{-2} \right).$$

Thus we only need to ensure that

$$\delta_\ell \, \varepsilon_\ell \leq 1, \quad \text{and} \quad \frac{1}{\delta_\ell} - \varepsilon_\ell \geq \frac{1}{\delta_u} + \varepsilon_u,$$

in order to get $\circledast$.

Chosen so that

One can now choose $A^{(0)} = 0$,

$$\delta_\ell = 1 \qquad \varepsilon_\ell = \frac{1}{\sqrt{d}}, \qquad \ell_0 = -\frac{n}{\varepsilon_\ell}, \quad \phi_{\ell_0}(0) = \varepsilon_\ell$$

$$\delta_u = \frac{\sqrt{d}+1}{\sqrt{d}-1}, \quad \varepsilon_u = \frac{1}{\sqrt{d}} \cdot \frac{\sqrt{d}-1}{\sqrt{d}+1} \qquad u_0 = \frac{n}{\varepsilon_u}. \quad \Phi_{u_0}(0) = \varepsilon_u.$$

Then,
$$\frac{1}{\delta_u} + \varepsilon_u = \frac{\sqrt{d}-1}{\sqrt{d}+1} \left(1 + \frac{1}{\sqrt{d}} \right) = 1 - \frac{1}{\sqrt{d}} = \frac{1}{\delta_\ell} - \varepsilon_\ell,$$

and
$$\frac{\lambda_{max}(A^{(nd)})}{\lambda_{min}(A^{(nd)})} \leq \frac{\frac{1}{\varepsilon_u} + d\,\delta_u}{-\frac{1}{\varepsilon_\ell} + d\,\delta_\ell} = \left(\frac{\sqrt{d}+1}{\sqrt{d}-1} \right)^2$$