Average difference inequality. Let $f : \Omega^n \to \mathbb{R}$ be s.t.

if $\quad f(y) - f(x) \leq \sum_{i=1}^{n} c_i(x)\, \mathbb{I}[x_i \neq y_i] \quad \forall x, y \in \Omega^n$, then

if $Z = f(X)$ and $\nu = \sup_x \sum_{i=1}^{n} c_i(x)^2$, where X is distributed according to a product distribution on Ω^n, then

$$\Pr[Z - E_p[Z] > t] \leq \exp\left(-\frac{t^2}{2\nu}\right), \quad \Pr[Z - E_p[Z] < -t] \leq \exp\left(-\frac{t^2}{2\nu}\right) \quad \forall\, t > 0.$$

(Here P is the distribution of X.)

Crucial difference from the bounded difference inequality:

$$f(y) - f(x) \leq \sum_{i=1}^{n} \underbrace{c_i}_{\text{no dependence on } X} \mathbb{1}[x_i \neq y_i] \qquad [\text{Bdd. diff.}]$$

Suppose $f(y) - f(x) \leq \sum_{i=1}^{n} c_i(X)\, \mathbb{I}[X_i \neq Y_i]$. As before,

$$E_Q f - E_P f = \underset{\substack{(X,Y)\sim\mathcal{C} \\ \mathcal{C}\text{: coupling} \\ \text{of } P, Q}}{E}[f(Y) - f(X)]$$

$$\leq \sum_{i=1}^{n} E[c_i(X)\, \mathbb{I}[X_i \neq Y_i]]$$

$$= \sum_{i=1}^{n} E[c_i(X)\, E[X_i \neq Y_i \mid X]]$$

$$= \sum_{i=1}^{n} E[c_i(X)\, P[X_i \neq Y_i \mid X]]$$

$$\leq \sum_{i=1}^{n} E_p[c_i(X)^2]^{1/2}\, E_{\mathcal{C}}[P_{\mathcal{C}}^2[X_i \neq Y_i \mid X]]^{1/2}.$$

$$\leq E_p\left[\sum_{i=1}^{n} c_i(X)^2\right]^{1/2} E_{\mathcal{C}}\left[\sum_{i=1}^{n} P_{\mathcal{C}}^2[X_i \neq Y_i \mid X]\right]^{1/2}$$

$$\leq \sqrt{2v} \quad \mathbb{E}_{\mathcal{C}}\left[\sum_{i=1}^{n} P_{\mathcal{C}}^{2}\left[X_i \neq Y_i \mid X\right]\right]^{1/2}$$

Thus we need to minimize over all couplings; and show that

$$\inf_{\mathcal{C}} \mathbb{E}\left[\sum_{i=1}^{n} \mathbb{P}[X_i \neq Y_i \mid X]^2 \right] \leq C_0 \, D(Q \| P)$$ for some constant C_0.

As before one can show that it is sufficient to handle the case $n = 1$.

Now, consider the optimal coupling from last time with density

$$C(x,x) = \min\{p(x), q(x)\}$$

$$C(x,y) = \frac{1}{\delta}\left(p(x) - q(x)\right)_+ \left(q(y) - p(y)\right)_+$$

where

$$\delta = \int (p(x) - q(x))_+ \, dx = \int (q(x) - p(x))_+ \, dx = \| p - q \|_{TV}.$$

We then have $P_C(X \neq Y \mid X)$

$$= \frac{(q(x) - p(x))_+}{p(x)} = \left(1 - \frac{q(x)}{p(x)}\right)_+$$

So, the l.h.s. is $\displaystyle\int p(x) \left(1 - \frac{q(x)}{p(x)}\right)_+^2 dx$

Recall $\displaystyle D(Q\|P) = \int p(x) . \frac{q(x)}{p(x)} \log\left(\frac{q(x)}{p(x)}\right) dx$