Miscellaneous topics

ε-nets :- In the last class we saw an application of ε-nets to proving concentration inequalites. Today we look at a more classical example.

Let $A \in \mathbb{R}^{kn \times n}$ be a matrix each entry of which is chosen uniformly at random from $N(0,1)$.

$\boxed{\text{Q :- What is } \|A\|_{2\to 2} = \sup_{\|x\|_2 \le 1} \frac{\|Ax\|_2}{\|x\|_2} \text{ ?}}$

Consider a fix x with $\|x\|=1$.

Claim :- $Ax \sim N(0, I_{kn})$.

Then we get

$$\Pr\left[\ \|Ax\|_2 \geq \sqrt{kn} + t\right] \leq \exp\left(-\frac{t^2}{2}\right)$$

We saw last time that there is a set S of $\left(1+\frac{2}{\varepsilon}\right)^n \leq \left(\frac{3}{\varepsilon}\right)^n$ points y with $\|y\| \leq 1$ which form an ε-net for $B_n(0,1)$ $[\varepsilon \leq 1]$

Thus,

$$\Pr\left[\exists x \in S \text{ s.t. } \|Ax\|_2 \geq \sqrt{kn} + t\right]$$
$$\leq \exp\left(-\frac{t^2}{2} + n\log\left(1+\frac{2}{\varepsilon}\right)\right)$$

But we want to bound the probability that

$$\exists y \in B_n(0,1) \text{ s.t. } \|Ay\|_2 \geq \sqrt{kn} + t$$

One can use a simple trick in this case. Suppose $\|A\|_{2\to 2} = M$. Then

$\exists\, y$, $\|y\| \leq 1$ s.t.

$$M \leqslant \|Ay\|_2 \leqslant \max_{x \in S} \|Ax\|_2 + M\varepsilon$$

$$\Rightarrow \quad M \leqslant \frac{1}{1-\varepsilon} \max_{x \in S} \|Ax\|_2 .$$

Thus taking $\varepsilon = 2$, we get

$$M \leqslant 2\left(\sqrt{kn} + t\right) \quad \text{w.p.} \geqslant 1 - \exp\left(-\frac{t^2}{2} + n \log 2\right)$$

Or, with $t = 2\sqrt{n}$,

$$\boxed{M \leqslant 2\left(\sqrt{k} + 2\right)\sqrt{n} \quad \text{w.p.} \geqslant 1 - \exp(-n).}$$

In particular, if $k = 1$, then $B = \frac{A}{\sqrt{n}}$ has

$$\|B\|_{2\to 2} \leqslant 6 \quad \text{w.p.} \geqslant 1 - \exp(-n).$$