Condition number

Recall that we settled on comparison of relative errors as a measure of stability. Formally, given a fn.

$$f : \mathbb{R} \to \mathbb{R}.$$

$$\text{RelError}(x) = \frac{|\tilde{x} - x|}{|x|}$$ where $\tilde{x}$ is a perturbed vesion of x.

$$\text{RelError}(f(x)) = \frac{|f(\tilde{x}) - f(x)|}{|f(x)|}.$$

So we have, roughly

$$\text{cond}^f(x, \tilde{x}) \approx \frac{\text{RelError}(f(x))}{\text{RelError}(x)}$$

But we expect this to make sense only if $\text{RelError}(x) \ll 1$

So we define

$$\text{cond}^f_\delta(x) = \sup_{\text{RelError}(x) \leq \delta} \frac{\text{RelError}(f(x))}{\text{RelError}(x)}$$

The condition number is defined as

$$\mathrm{cond}^f(x) := \lim_{\delta \downarrow 0} \mathrm{Cond}^f_\delta(x) = \lim_{\delta \downarrow 0} \sup_{\mathrm{RelError}(x) \le \delta} \frac{\mathrm{RelError}(f(x))}{\mathrm{RelError}(x)}$$

The limit definition is useful for analysis. For applications to algorithms, we will use the following consequence of this definition.

$$\mathrm{RelError}(f(x)) = \mathrm{RelError}(x) \cdot \mathrm{cond}^f(x) + o(\mathrm{RelError}(x)).$$

<u>Note</u>:- Note again that $\mathrm{cond}^f(x)$ is a property of the function f, <u>not</u> of any particular algorithm for computing f.

<u>Interpretation of the condition number</u>

– Suppose the correct input x is 'perturbed' [due to floating point rounding, perhaps] so that the available input is $\tilde{x} = x(1+\delta)$ where $|\delta|$ is small.

Then, one has

$$\frac{|f(\tilde{x}) - f(x)|}{|f(x)|} \simeq \delta \cdot \left[\mathrm{cond}^f(x) + o(1)\right],$$

and we cannot expect <u>any</u> algorithm to have a better relative error in the output

Some examples

(1) $f(x) = x$. $\quad \text{cond}^f(x) = 1$

(2) $f(x) = x^2$: $\displaystyle \lim_{\delta \downarrow 0} \sup_{|\delta'| \leq \delta} \frac{|(x+\delta' x)^2 - x^2| / |x|^2}{|\delta'|}$

$$= \lim_{\delta \downarrow 0} \sup_{|\delta'| \leq \delta} |2 + \delta'| = 2.$$

$$\text{cond}^f(x) = 2$$

(3) $f(x) = e^x$: $\displaystyle \frac{|e^{x(1+\delta)} - e^x| |e^{-x}|}{|\delta|}$

$$= \frac{1}{|\delta|} |e^{\delta x} - 1|$$

$$= \frac{1}{|\delta|} |\delta x + O(\delta^2)|$$

$$\leq |x| + O(\delta)$$

$\Rightarrow \text{cond}^f(x) \leq |x|$ (Actually equality holds.)

Backward analysis

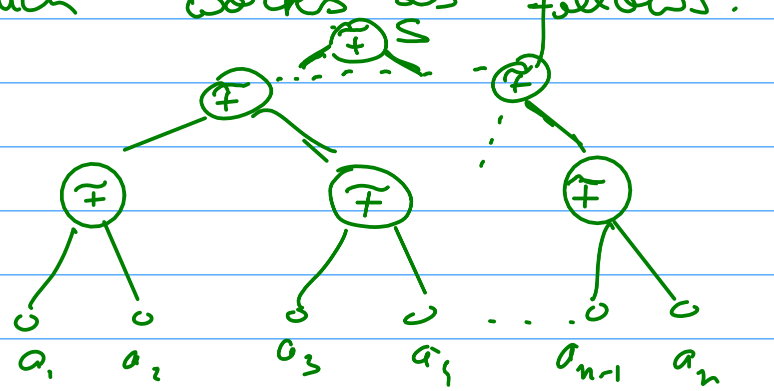
Suppose A is an algorithm for computing a function f. Suppose that one can prove that it has the following backward stability property: on input x, the algorithm outputs $f(x + e_x)$, s.t. $\|e_x\| \leq \delta \|x\|$ for some fixed small constant δ.

We then claim that A is also forward stable in the following sense:

$$\frac{\|A(x)-f(x)\|}{\|f(x)\|} \leq \delta \, \mathrm{cond}^f(x) + o(\delta).$$

[Exercise, from definition.

Example: Consider $f(a_1, a_2, \ldots a_n) = \sum a_i$. Let A be an algorithm for computing this on a floating point machine with machine epsilon ε_{mach}, which works as follows:



'$\tilde{+}$' refers to addition with rounding: $a \tilde{+} b = (a+b)(1+\delta)$ when $|\delta| \leq \varepsilon_{mach}$.

It can then be shown that

$$A(a_1, \ldots a_n) = f(\tilde{a}_1, \tilde{a}_2, \ldots \tilde{a}_n)$$

where $|\tilde{a}_i - a_i| \leq \varepsilon_{mach} |a_i| (\lceil \log_2 n \rceil + 2)$, provided $\varepsilon_{max} (\lceil \log_2 n \rceil + 2)^2 < 1$