Condition number of matrix inversion [Turing/von-Neumann]

Consider the function

$$f(A) = A^{-1}.$$

We will analyze the condition number of this function. It will turn out to have several applications: most notably the analyses of the gradient descent and the conjugate gradient descent methods.

We first need to fix what norm we will use to measure errors when the underlying objects are matrices. The right answer turns out to be the operator norm, defined for a matrix A as follows:

$$\|A\| = \|A\|_{2\to 2} := \sup_{\|x\|_2 \le 1} \|Ax\|_2$$

Ex: Verify that this is a norm: [Convex, linear w.r.t. multiplication by constant, positive semi-definite, and attaining the value 0 only when $A = 0$].

The operator norm is also <u>submultiplicative</u>

$$\|AB\| \le \|A\| \cdot \|B\| \quad \text{(why?)}.$$

Now we define the condition no. as:

$$\kappa_A := \mathrm{cond}^f(A) = \lim_{\delta \downarrow 0} \sup_{E : \frac{\|E\|}{\|A\|} \le \delta} \frac{\frac{\|(A+E)^{-1} - A^{-1}\|}{\|A^{-1}\|}}{\frac{\|A+E-A\|}{\|A\|}}.$$

<u>Thm</u>:- $\kappa_A = \|A\| \cdot \|A^{-1}\|$.

<u>Pf</u>:- We first show that

$$\kappa_A \le \|A\| \cdot \|A^{-1}\|.$$

We first note

$$(A+E)^{-1} = \left((I + EA^{-1})A\right)^{-1} = A^{-1}(I+EA^{-1})^{-1}.$$

$$= A^{-1}(I - EA^{-1}) + D$$

where $\|D\| = O(\|E\|^2)$.

Thus, $\|(A+E)^{-1} - A^{-1}\| = \|-A^{-1}EA^{-1} + D\|$

$$\le \|A^{-1}\|^2 \|E\| + O(\|E\|^2). \quad \text{— (1)}$$

So,

$$\kappa_A \leqslant \lim_{\delta \downarrow 0} \sup_{E : \|E\| \leqslant \delta} \frac{\dfrac{\|A^{-1}\|^2 \|E\| + O(\|E\|^2)}{\|A^{-1}\|}}{\dfrac{\|E\|}{\|A\|}}$$

$$= \lim_{\delta \downarrow 0} \left(\|A\| \, \|A^{-1}\| + O(\delta^2) \right)$$

$$= \|A\| \, \|A^{-1}\|.$$

Thus, $\kappa_A \leqslant \|A\| \cdot \|A^{-1}\|$.

We now show that this is an equality.

Claim:- If E is s.t. $\|A^{-1} E A^{-1}\| = \|A^{-1}\|^2 \|E\|$, then ① holds with equality. [Ex.]

Now, let y be s.t.

$$\|A^{-1} y\| = \|A^{-1}\| ; \ \|y\| = 1$$

(Such a y exists, [Ex.])

Let $x = \dfrac{\|A^{-1} y\|}{\|A^{-1}\|}$, and $B = y x^T$.

Claim:- Setting $E = \delta B$ is sufficient. [Why?]

Condition number of lin. system solving

Consider

$$Ax = b. \qquad \underline{\varphi(A,b) = A^{-1}b.}$$

$$A \in \mathbb{R}^{n\times n}, \quad b \in \mathbb{R}^{n}.$$

For perturbed versions $\tilde{A}$, $\tilde{b}$ of A, b, we set

$$R := \operatorname{RelError}(A,b) = \max\left\{ \frac{\|A-\tilde{A}\|}{\|A\|}, \frac{\|b-\tilde{b}\|}{\|b\|} \right\}.$$

We want to estimate

$$\operatorname{cond}_{\varphi}(A,b):$$

Set $x = \varphi(A,b) = A^{-1}b$, $\tilde{x} = \tilde{A}^{-1}\tilde{b}$

$$\operatorname{RelError}(\varphi(A,b)) = \frac{\|\tilde{x}-x\|}{\|x\|}.$$

Let $\tilde{A} = A + E$, $\tilde{b} = b + e$ so that

$$\|E\| \le R\|A\|, \quad \|e\| \quad R\|b\|.$$

Now

$$\begin{aligned}
\tilde{x} &= (A+E)^{-1}(b+e) \\
&= ((I+EA^{-1})A)^{-1}(b+e) \\
&= A^{-1}(I+EA^{-1})^{-1}(b+e)
\end{aligned}$$

Now when $\|EA^{-1}\| < 1$,

$$(I+EA^{-1})^{-1} = I - EA^{-1} + (EA^{-1})^2 - (EA^{-1})^3 \cdots$$
$$= I - EA^{-1} + D$$
where $\|D\| = O(R^2)$.

So,
$$\tilde{x} = (A^{-1} - A^{-1}EA^{-1})(b+e) + O(R^2)$$
$$= x + A^{-1}e - A^{-1}EA^{-1}b + A^{-1}EA^{-1}e + O(R^2)$$

But $\|AEA^{-1}e\| = O(R^2)$, so we have,

$$\|\tilde{x}-x\| \le \|A^{-1}e - A^{-1}Ex\| + O(R^2) \quad — \ (*)$$
$$① \le \|A^{-1}e\| + \|A^{-1}Ex\| + O(R^2)$$
$$② \le \|A^{-1}\|\,\|e\| + \|A^{-1}\|\,\|E\|\,\|x\| + O(R^2)$$
$$③ \le R\,\|A^{-1}\|\,\|b\| + R\,\|A^{-1}\|\,\|A\|\,\|x\| + O(R^2)$$
$$\Rightarrow \frac{\|\tilde{x}-x\|}{\|x\|} \le R\left(\kappa_A + \frac{\|A^{-1}\|\cdot\|b\|}{\|A^{-1}b\|}\right) + O(R^2)$$

Now, by definition of R

$$\text{Cond}_\psi(A,b) = \lim_{R \downarrow 0} \frac{\frac{\|\tilde{x}-x\|}{\|x\|}}{R} \le \kappa_A + \frac{\|A^{-1}\|\cdot\|b\|}{\|A^{-1}b\|}$$

To probe equality, we need to choose the perturbations E and e carefully so that inequalities

(1), (2) and (3) above hold with equality.

Let v be a unit vector such that $\|A^{-1}v\| = \|A^{-1}\|$. Let $y = \frac{A^{-1}v}{\|A^{-1}\|}$ be the unit vector in the direction of $A^{-1}v$, and $z = \frac{x}{\|x\|}$ the unit vector in the direction of x. Let B be the matrix $-vz^T$. Now note that

$$A^{-1}v = \|A^{-1}\| y$$

$$A^{-1}Bx = -\|A^{-1}\| \, \|x\| \, y$$

Let

$$e = R\|b\| v \quad , \quad E = R\|A\| B.$$

Then, from (*),

$$\|\tilde{x} - x\| \geq \|A^{-1}e - A^{-1}Ex\| - O(R^2)$$

$$= R \big\| \|A^{-1}\| \|b\| y + \|A\| \|A^{-1}\| \|x\| y \big\| - O(R^2)$$

$$= R\left(\|A^{-1}\| \|b\| + \|A\| \|A^{-1}\| \|x\|\right) - O(R^2)$$

since $\|y\| = 1$.

$$\Rightarrow \frac{\|\tilde{x} - x\|}{\|x\|} \geq R\left(\kappa_A + \frac{\|A^{-1}\| \|b\|}{\|A^{-1}b\|}\right) - O(R^2)$$

so that

$$\operatorname{cond}_\varphi(A, b) \geq \kappa_A + \frac{\|A^{-1}\| \, \|b\|}{\|A^{-1}b\|}.$$

$\square$