Interlacing :- Let $f_1, \ldots \ldots f_n$ be degree d univariate real polynomials with real roots and positive leading coefficient. Suppose that they have a common interlacing: ie a sequence of numbers

$$\alpha_0 \geqslant \alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \quad \cdots \quad \geqslant \alpha_d .$$

st. if $\beta_{i,1} \geqslant \beta_{i,2} \geqslant \beta_{i,3} \cdots \geqslant \beta_{i,d}$ are the d roots of f_i (counted with multiplicity).

then

$$\alpha_0 \geqslant \beta_{i,1} \geqslant \alpha_1 \geqslant \beta_{i,2} \geqslant \alpha_2 \geqslant \cdots\cdots \geqslant \alpha_{d-1} \geqslant \beta_d \geqslant \alpha_d .$$

Suppose that $f = \sum_{i=1}^{n} f_i$ has largest real root at most γ. Then $\exists\, i \in [n]$ such that the max root $\beta_{i,1}$ of f_i is at most γ.

Remark :- Define $I_j^i := [\beta_{i,j+1}, \beta_{i,j}]$ for $j = 1$ to $d-1$

$$I_0^i = [\beta_{i,1}, \infty)$$

$$I_d^i = (-\infty, \beta_{i,d}].$$

Then $(f_i)_{i \in [n]}$ have a common interlacing iff and only if $\forall\, j \in [0, d]$, $\bigcap_{i=1}^{n} I_j^i$ is non-empty.

Proof :- For all 'large enough' x. $f_i(x) > 0$. for every $i \in [n]$. In particular $f_i(\alpha_0) \geq 0, \forall i$ (and we can choose α_0 s.t. $f_i(\alpha_0) > 0 \; \forall i$.)

So, $f(\alpha_0) > 0$. — (1).

Now $f_i(\alpha_1) \leqslant 0 \quad \forall \; i \in [n]$.

[If $f_i(\alpha_1) > 0$ and $f(\alpha_0) > 0$ then the no. of roots of f_i in $[\alpha_1, \alpha_0]$ is even. (counting with multiplicitities. But this contradicts that there is exactly one root in $[\alpha_1, \alpha_0]$, counting multiplicities.]

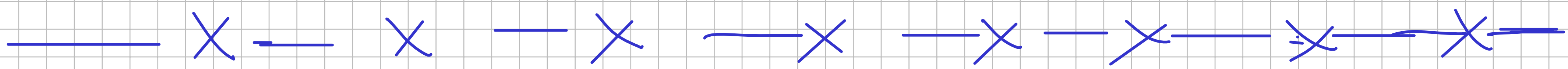
$\Rightarrow \quad f(\alpha_1) \leq 0.$

So, $\quad f(\alpha_1) \leq 0 \qquad f(\alpha_0) > 0 \qquad \underline{\alpha_0 > \alpha_1}$ (1)

So, the max root γ of f must be at least α_1. (because (1) implies that f has a root in $[\alpha_1, \alpha_0]$).

Now, $f(\gamma) = 0 \Rightarrow$ So there must be some i for which $f_i(\gamma) \geq 0$. But $\gamma \geq \alpha_1$, $f_i(\alpha_1) \leq 0$, so f_i must have a root in $[\alpha_1, \gamma] \Rightarrow \underline{\beta^i_i \leq \gamma}$.

So $\text{max-root}(f_i) \leq \text{max-root}(f)$.



So how do we prove common interlacing?

Lemma ⊛ :- Let f and g be degree d polynomials with positive leading coefficient. Suppose that

$$p \cdot f + q g \quad \forall\, p, q \geq 0$$

is real rooted. Then f and g have a common-interlacing.

[Chudnovsky and Seymour, 04] proof.

[Independence polynomial of claw-free graphs]

Lemma :- Suppose f, g are as in Lemma ($\ast$). Let $a < b$ be points where f and g are both non-zero and have the same sign: $f(a)g(a) > 0$ and $f(b)\,g(b) > 0$. Define, for any polynomial p and any point $t \in \mathbb{R}$ $n_p(t) := \{$ no. of roots of p in $[t, \infty)$, counted with multiplicity$\}$

Then,

$$n_f(b) - n_f(a) = n_g(b) - n_g(a).$$

Proof :- $n_f(b) - n_f(a) =$ no. of roots of f in $[a, b)$

$=$ no. of roots of f in (a, b) $\forall\ f(a) \neq 0.$

Similarly

$$n_g(b) - n_g(a) = \text{no. of roots of } g \text{ in } [a, b)$$
$$= \text{no. of roots of } g \text{ in } (a, b) \text{ because } g(a) \neq 0.$$

Consider $p_\lambda(x) = \lambda f(x) + (1-\lambda) g(x)$. $\lambda \in [0,1]$.

p_λ is real-rooted. The roots of p_λ are continuous functions of λ (this does not depend upon p_λ being real rooted).

Note that $\forall \lambda \in [0,1]$, $p_\lambda(a) \neq 0$, $p_\lambda(b) \neq 0$. So by the continuity of roots all p_λ have the same no. of roots in (a, b) (as p_λ are also real-rooted). ∎

<u>Obv</u> :- If f and g have degree d and positive leading coefficient and

$$|n_f(x) - n_g(x)| \leq 1 \quad \forall x$$

then f and g have a common interlacing.

$n_f(x) =$ no. of roots of f that are at least x

$=$ no. of roots of f in $[x, \infty)$.

(counted with multiplicity)

Pf :- (Sketch) Just list the roots in decreasing order and label them as 'f' and 'g' (based on which polynomial they come from) alternating between f and g

wherever possible. Now move left from $+\infty$ and set a interlacing point everytime the difference becomes zero.

Final lemma :- If f, g are as in Lemma $(*)$ then $\forall x$, $|n_f(x) - n_g(x)| \leq 1$.

Proof :- Induction on degree d :

(1) $d = 1 \Rightarrow$ Trivially true, $f = c_1(x - r_1)$ $g = c_2(x - r_2)$
$c_1, c_2 \geq 0$. So $\forall x$ $|n_f(x) - n_g(x)| \leq 1$
$r_1, r_2 \in \mathbb{R}$.

Assume that the statement is true whenever f, g have common degree $\leq d-1$. Now,

Let f, g be of degree d. And suppose if possible that there exists x_0 s.t.

$$n_f(x_0) - n_g(x_0) \geq 2. \quad \text{—} ① \quad \text{(Relabel } f \text{ \& } g \text{ if necessary).}$$

Let x_0 be the largest value that satisfies ①. Then x_0 must be a root of f.

Let us now consider f' and g', which have degree $\underline{d-1}$ and also satisfy ⊛ so by $\underline{\text{induction hypothesis}}$,

$$|n_{f'}(x_0) - n_{g'}(x_0)| \leq 1 \quad \text{—} ②$$

But $n_{f'}(x_0) = n_f(x_0) - 1$, $n_{g'}(x_0) \leq n_g(x_0)$

So $n_f(x_0) - n_g(x_0) \leq \underbrace{n_{f'}(x_0) - n_g(x_0) + 1}_{\leq 2}$. — ③

So, 1, 2 and 3 imply that the only possibility remaining is

$$n_f(x_0) - n_g(x_0) = 2.$$

There is some $y_0 > x_0$ s.t. both f and g are positive $\forall\ y \geq y_0$. In particular $n_f(y_0) - n_g(y_0) = 0$.

<u>Goal</u> :- <u>If we could now choose a point $x_1 < x_0$</u>
<u>$x_1 = x_0 - \varepsilon$,</u> for ε small enough s.t. $f(x_1) \neq 0$ $g(x_1) \neq 0$
and $n_f(x_1) - n_g(x_1) = 2$, then we would get that $f(x_1)$ and $g(x_1)$ have the same sign (because they

have the same parity of roots in (x_1, y_0). But then we have $f(x_1)\, g(x_1) > 0$ $f(y_0)\, g(y_0) > 0$ and

$$n_f(x_1) - n_g(x_1) = 2$$

$$n_f(y_0) - n_g(y_0) = 0.$$

This would be a contradiction to the fact that $n_f(x_1) - n_g(x_1) = n_f(y_0) - n_g(y_0)$ as f, g satisfy condititions of $(*)$ and have

$$f(x_1)\, g(x_1) > 0 \qquad f(y_0)\, g(y_0) > 0.$$

The only problem in implementing this strategy is that x_0 could be a root of g as well.

But this is fixed by factoring out the Common roots of f and g. This can potentially decrease the degree of f and g, but by the same amount, but (1) still keeps conditions of Lemm (*) intact (2) keeps the value of $n_f(x) - n_g(x)$ unchanged $\forall\, x \in \mathbb{R}$.

- So if x_0 is a common root of f & g we can just induction hypothesis after removing the common factor $x - x_0$. Otherwise if x_0 is a root only of f, we apply the strategy of choosing $x_1 < x_0$ as before. $\blacksquare$.