Fully Dynamic Maximal Matching in $O(\log n)$ update time

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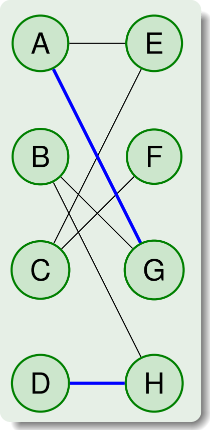
Outline

1. **Introduction**
   - The Problem
   - A Simple Agorithm
2. **$\sqrt{n}$ algorithm**
   - The overview of the approach
   - Overview of the analysis
   - Algorithm
3. **From $\sqrt{n}$ to $\log n$**
   - Speeding up the algorithm
4. **Open Problem**

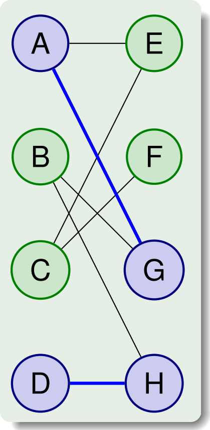
Some Definitions

- A matching in a graph is a set of edges M such that no two edges in M share a common endpoint.



Some Definitions

- A matching in a graph is a set of edges M such that no two edges in M share a common endpoint.
- Matched vertex



Some Definitions

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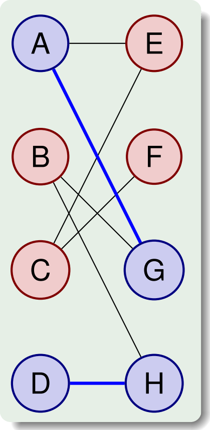
- Matched vertex
- Free vertex



The Problem

A matching is maximal if for each vertex v:

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Maintain maximal matching in a dynamic graph

Expectation from the algorithm

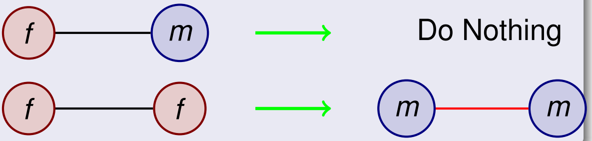
Update time should be polylog(n)

Previous Work

- Ivkovic and Llyod(1994) - $O((n+m)^{0.7072})$
- Onak and Rubinfeld(2010) gave a c-approximation of maximum matching in $O(\log^2 n)$ update time.

A Naive Approach

Insertion of an edge



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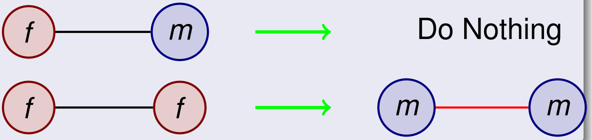

Deletion of an edge

A Naive Approach

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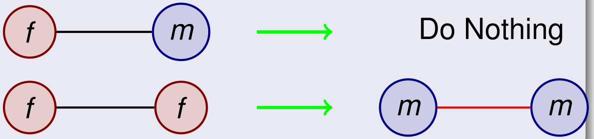

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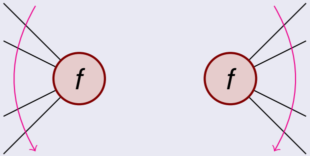

Search neighborhood of both vertex for free vertex

A Naive Approach

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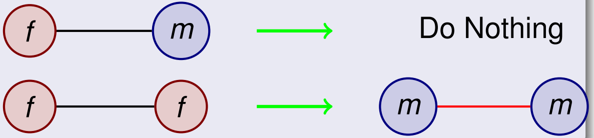

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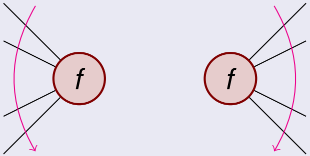

Search neighborhood of both vertex for free vertex

- Insertion = $O(1)$
- Deletion = $O(n)$

The difficulty

- Deletion of a matched edge
- Handling high degree vertex

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- Handling high degree vertex

Possible ways to solve

- Make sure that high degree vertex are always matched
- Make sure that a matched edge is deleted rarely

- Partition the vertices into two buckets(level 1 and 0) such that most of the vertices have high "degree" when they come to level 1
- The partition is dynamic and the vertices may move from level 1 and level 0
- Maintain the following invariant
  - The vertex at level 1 are always matched
  - The vertex at level 0 has degree $< \sqrt{n}$ in $G[V_0]$ and each free vertex at this level has all its neighbors matched

The overview of the approach

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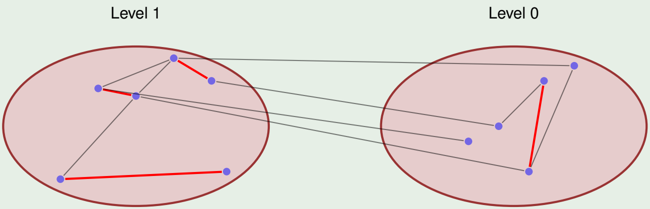

Notion of ownership

Each edge present in the graph will be owned by one or both of its end points as follows

- If both the end points are at level 0, then it is owned by both the endpoints
- If only one endpoint is at level 1, then it owns the edge
- If both the end points are at the same level, we can break the tie arbitrarily

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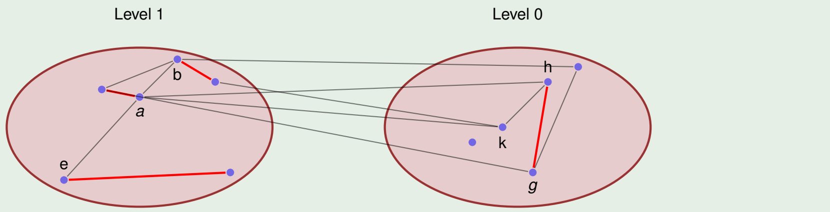

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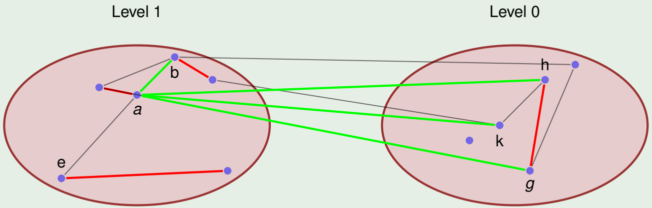

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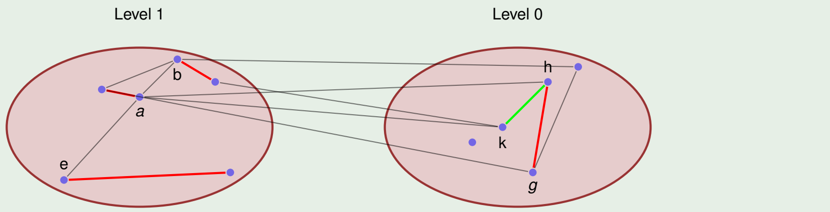
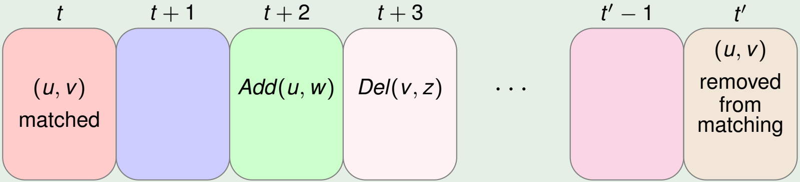


Notion of matched epoch

Epoch of (u, v) is the maximal continuous time period for which it remains in the matching.



Epoch of Level 0

- Inv2:The vertex at level 0 has degree $\sqrt{n}$ in $G[V_0]$ and each free vertex at this level has all its neighbors matched

Start of the epoch

But...what if u has more than $\sqrt{n}$ edges in $G[V_0]$ after edge insertion?

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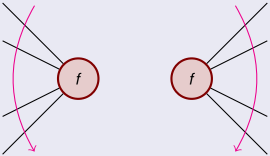

Search only the edges whose other endpoint are at level 0

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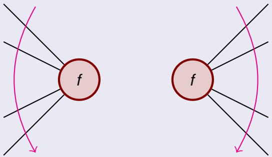


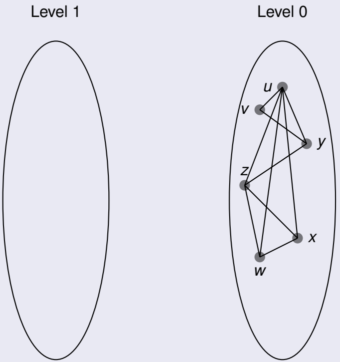
Search only the edges whose other endpoint are at level 0

- Start = $O(1)$
- End = $O(\sqrt{n})$

Epoch at level 1: Start

- Inv2:The vertex at level 0 has degree $\sqrt{n}$ in $G[V_0]$ and each free vertex at this level has all its neighbors matched

Invariant 2 does not hold for vertex u



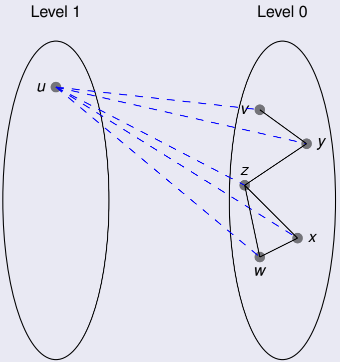
Epoch at level 1: Start

- Inv2:The vertex at level 0 has degree $\sqrt{n}$ in $G[V_0]$ and each free vertex at this level has all its neighbors matched

$\longrightarrow$ Vertex u moves to level 1

$\longrightarrow$ Make u the *owner* of all its edges at level 0

$\longrightarrow$ Find a random edge from the owned edges, say (u, v)



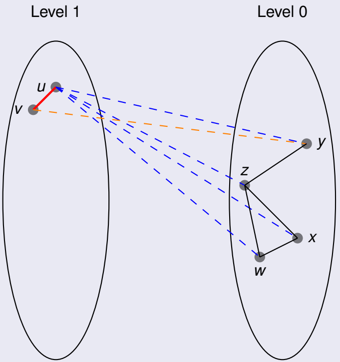
Epoch at level 1: Start

- Inv2:The vertex at level 0 has degree $\sqrt{n}$ in $G[V_0]$ and each free vertex at this level has all its neighbors matched

$\longrightarrow$ Add (u, v) to M

$\longrightarrow$ Move v to level 1

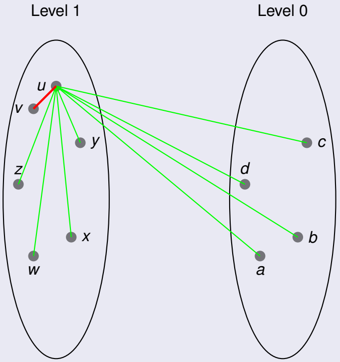
$\longrightarrow$ Make v the *owner* of all its adjacent edges



Start of an epoch at level 1 = $O(\sqrt{n})$

Epoch at level 1: End

Edge (u, v) is deleted

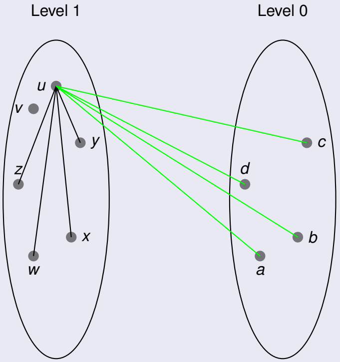


Epoch at level 1: End

$\longrightarrow$ Give up the ownership of the edges at level 1

$\longrightarrow$ If u is still the owner of $\geq \sqrt{n}$ edges

the the procedure is same as in the previous slide



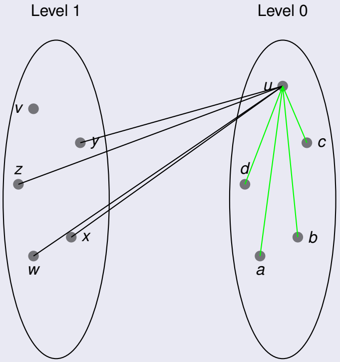
Epoch at level 1: End

⟶ Else u moves to level 1 and starts level 0 epoch there

⟶ But the degree of vertex a in $G[V_0]$ increses by 1

and may move to level 1

⟶ All such vertex move up and start a epoch at level 1



End of an epoch at level 1 = $O(n)$

Epochs	Start	End	Total cost	Total number of Epochs	Total computation cost
Level 0	$O(1)$	$O(\sqrt{n})$	$O(\sqrt{n})$	T	$O(T\sqrt{n})$
Level 1	$O(\sqrt{n})$	$O(n)$	$O(n)$		

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The algorithm has $O(\sqrt{n})$ update time.

Balance

In the two level algorithm, we define a threshold $\alpha(n)$ for a vertex to move from level 0 to level 1

- The update time at level 0 is $O(\alpha(n))$
- The update time at level 1 is $O(n/\alpha(n))$
- Both the update time are same when $\alpha(n) = \sqrt{n}$

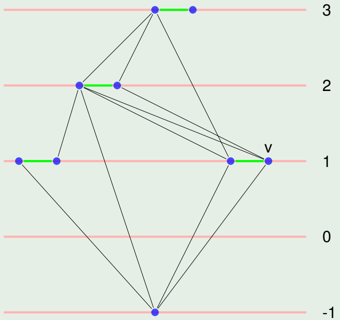
Speeding up the algorithm

- Try to minimize the gap between the number of edges a vertex can own in an epoch and the number of edges it owned at the moment it created the epoch
- This ratio is $\sqrt{n}$ in 2-level algorithm

An overview of the log n-level algorithm

- Maintain log n levels
- When a vertex creates an epoch at level i, it would own at least 2^i edges, and during the epoch it will be allowed to own at most 2^{i+1} edges
- The ratio is a constant
- In implementing these ideas, an extra factor of $O(\log n)$ comes up due to the log n level hierarchy

Example



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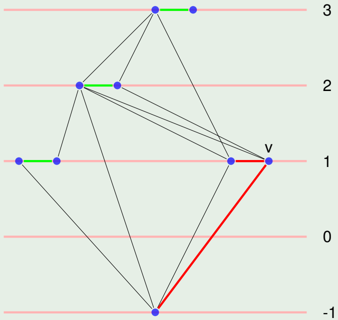
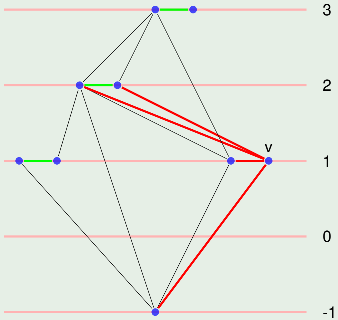


The number of edges v can own if it rises to level 2 = 2 $< 2^2$

Example



The number of edges v can own if it rises to level 3 = 4 $< 2^3$

Open Problem

- There exists an algorithm for maximal matching in $O(\log n)$ update time but is there a algorithm which maintains $c-approximation$ of maximum matching where $c < 2$
- Is there any combinatorial algorithm which maintains maximum matching in $o(m)$ time

Questions?