Communication Complexity of Gap Hamming Problem à la Sherstov

Board talk based on

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The Gap Hamming Problem

Alice: $x \in \{0,1\}^n$
Bob: $y \in \{0,1\}^n$

$$\text{Answer} = \begin{cases} \text{Yes} & \text{if } d(x,y) > \frac{n}{2} + \sqrt{n} \\ \text{No} & \text{if } d(x,y) < \frac{n}{2} - \sqrt{n} \\ \text{Don't care} & \text{Otherwise} \end{cases}$$

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More convenient: $x, y \in \{-1,+1\}^n$.

$$\text{Answer} = \begin{cases} -1 & \text{if } \langle x,y \rangle \leq -\sqrt{n} \\ +1 & \text{if } \langle x,y \rangle \geq \sqrt{n} \\ \text{Don't care} & \text{Otherwise.} \end{cases}$$

Gap Orthogonality (Sherstov)

Alice: $x \in \{-1,+1\}^n$.
Bob: $y \in \{-1,+1\}^n$.

$$\text{Answer} = \begin{cases} -1 & \text{if } |\langle x,y\rangle| \le \sqrt{n}/8 \\ +1 & \text{if } |\langle x,y\rangle| \ge \sqrt{n}/4 \\ \text{Don't care} & \text{Otherwise} \end{cases}$$

Concentration

Theorem (Talagrand)

Let V be a linear subspace of $\mathbb{R}^n$. There is a constant $c > 1$ such that for a random $x \in \{-1, +1\}^n$ and all $t > 0$,

$$\Pr[|\|\pi_V\, x\| - \sqrt{dim V}| > t + c] < 4 \exp\left(-\frac{t^2}{c}\right).$$

Almost orthogonal vectors

Lemma

Fix $A \subseteq \{-1,+1\}^n$ *sufficiently large* ($|A| > 2^{(1-\alpha)n}$). *Then, one can find* $m > k/10$ *elements* $x_1, x_2, \ldots, x_m \in A$ *such that*

$$\|\pi_{S_{i-1}}(x_i)\| \leq \frac{\sqrt{n}}{3}.$$

Projection on to almost orthogonal vectors

Lemma

Fix vectors $x_1, x_2, \ldots, x_m$ are almost orthogonal. Then,

$$\Pr_{y \in \{-1,+1\}^n}\left[\forall i \ \left|\langle y, x_i\rangle\right| \le \frac{\sqrt{n}}{4}\right] \le \exp(-\Omega(m)).$$

No large red rectangle

Theorem

Let $\gamma > 0$ be small enough. Let $R = A \times B$ be a rectangle. Suppose

$$\Pr_{(x,y)\in R}\left[|\langle x, y\rangle| > \frac{\sqrt{n}}{4}\right] \leq \gamma.$$

Then,

$$\mu(R) = 4^{-n}|A||B| < \exp(-\Omega(n)).$$