Rank-1 Two Player Games: A Homeomorphism and a Polynomial Time Algorithm

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Joint work with Bharat Adsul, Jugal Garg and Milind Sohoni

Outline

- Games, Nash equilibrium and history
- Two player finite games
- Known results
  - Rank and tractability
- Rank-1 games

Games

- Set of players - rational, intelligent, selfish.
- Each with a set of strategies - finite or infinite.
- Payoffs - preference over outcome is described through payoffs.
- Equilibrium - State from where no player gains by unilateral deviation.

Example: Rock-Paper-Scissor

Two players, say Row and Column.

Each with three strategies - R, P and S.

	R	P	S
R	0,0	-1,1	1,-1
P	1,-1	0,0	-1,1
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Finite games have finitely many players each with finitely many strategies.

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Equilibrium?

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- **John Nash (1950)** - Existence of equilibrium in finite games.
  - Proof is through Brouwer's fixed point theorem, hence highly non-constructive.
- Irrational NE in a 3-player game (3-Nash) (Nash, 1951).

Computational Results

- Lemke-Howson algorithm (1964) for exact 2-Nash - Exponential.
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- Papadimitriou (1992) defined PPAD.
  - Guaranteed existence but difficult to compute.
  - Contains Sperner's lemma, approximate fixed point, exact 2-Nash.
  - Approximate fixed point is PPAD-hard.

Contd.

- PPAD-hardness
  - Daskalakis, Goldberg and Papadimitriou (2006) - approximate 3-Nash.
  - Chen and Deng (2006) - exact 2-Nash.
  - Chen, Deng and Teng (2006) - approximate 2-Nash.

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- NP-hardness for 2-Nash: existence of two NE, NE with x payoff, NE with t non-zero strategies, etc.
- Dantzig (1963): mini-max strategies equivalent to linear primal-dual solution.
  - Zero-sum games can be solved efficiently.

Structural Results

- Shapley (1974) - oddness for 2-Nash.
- Shapley's index theory assigns a sign to a NE.
  - $|\text{NE with } +1 \text{ index}| = 1 + |\text{NE with } -1 \text{ index}|$.
  - Puts the NE computation problem in $\mathcal{PPAD}$.

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  - Puts the NE computation problem in $\mathcal{PPAD}$.
- Kohlberg and Mertens (1986) - Homeomorphism between (finite) game space and its NE correspondence.
  - Extends oddness and index results.
  - Existence and characterization of stable NE.
  - Validates the homotopy based NE computation methods.

2-player (Bimatrix) Games: Rank and Tractability

- Strategy sets: $S_1 = \{1, \ldots, m\}$ and $S_2 = \{1, \ldots, n\}$.
- Payoff matrices A and B (in $\mathbb{R}^{m \times n}$).
  - S_1 is the set of rows and S_2 is the set of columns.

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- Zero-sum $\equiv$ rank-0 - an LP captures all the NE.
- **Rank-1: No polynomial time algorithm was known.**
  - Difficulty: Disconnected NE set. Reduces to solving rank-1 QP (known to be NP-hard in general).

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- Efficient algorithm for an exact NE in rank-1 games - Settles an open question by Kannan and Theobald (2007).
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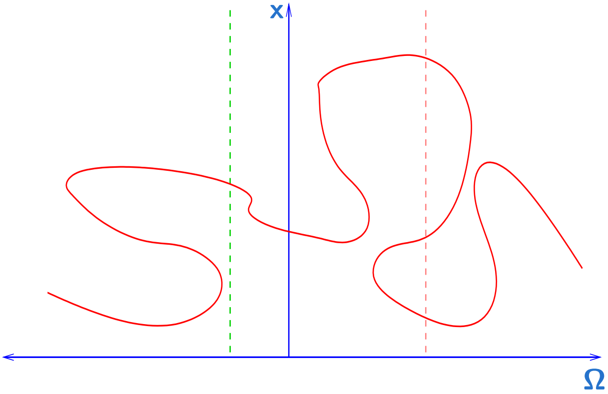
  - New proofs for existence, oddness and index theorem.

Based on the STOC'11 paper.

Homeomorphism

Two spaces are homeomorphic if they are topologically identical (#components, cross intersections, holes, ...)



Not clear if homeomorphism preserve subspaces.

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- Example: $y = (0.5, 0.3, 0.2)^T$

$$A \cdot y = \begin{matrix} R \\ P \\ S \end{matrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.5 \\ 0.3 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.3 \\ -0.2 \end{bmatrix}$$

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- **Best Response (BR):** P
- **BR:** $i \in S_1$ s.t. $A_i \cdot y = \max_{k \in S_1} A_k \cdot y$

Contd.

- Payoff from $x \in \Delta_1$ is a conv. comb. of $A_i y$'s.
  - x maximizes payoff iff $\forall i \in S_1,\ x_i > 0 \Rightarrow i$ is a BR to y.

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Nash Equilibrium: No player gains by unilateral deviation.

$$(x, y) \in \Delta_1 \times \Delta_2, \quad \begin{array}{lll} \forall i \in S_1,\ x_i > 0 & \Rightarrow & A_i \cdot y = \max_k A_k \cdot y \\ \forall j \in S_2,\ y_j > 0 & \Rightarrow & x^T \cdot B^j = \max_k x^T \cdot B^k \end{array}$$

Best Response Polyhedra, Fully-labeled and NE

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Best Response Polyhedra (BRPs) (Assume non-degeneracy)

$$P = \{(y, \pi_1) \mid A_i y - \pi_1 \le 0, \forall i \in S_1; \quad y_j \ge 0, \quad \forall j \in S_2;\ \Sigma_{j \in S_2} y_j = 1\}$$
$$Q = \{(x, \pi_2) \mid \quad x_i \ge 0, \quad \forall i \in S_1;\ x^T B^j - \pi_2 \le 0, \forall j \in S_2;\ \Sigma_{i \in S_1} x_i = 1\}$$

Where π_i captures the best payoff to player i.

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NE iff fully-labeled. Only vertex pairs.

Note: $x_i(A_i y - \pi_1) \le 0, \forall i; \quad y_j(x^T B^j - \pi_2) \le 0, \forall j$

QP for NE

Note:

- $x^T Ay - \pi_1 \leq 0; \quad x^T By - \pi_2 \leq 0$
- **NE iff** $x^T Ay - \pi_1 = 0$**;** $x^T By - \pi_2 = 0$ (fully-labeled).

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- **NE iff** $x^T Ay - \pi_1 = 0$**;** $x^T By - \pi_2 = 0$ (fully-labeled).

QP:

$$\begin{aligned} \max: \quad & x^T(A+B)y - \pi_1 - \pi_2 \\ & (y, \pi_1) \in P, \ (x, \pi_2) \in Q \end{aligned}$$

Optimal value is always zero.

Solving a Rank-1 Game (A, B)

$rank(A+B) = 1 \Rightarrow A + B = \alpha \cdot \beta^T,\ \alpha \in \mathbb{R}^m,\ \beta \in \mathbb{R}^n.$

$$\begin{aligned} \max: \quad & (x^T\alpha)(\beta^T y) - \pi_1 - \pi_2 \\ & (y, \pi_1) \in P,\ (x, \pi_2) \in Q \end{aligned}$$

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Consider B as $-A + \alpha \cdot \beta^T$ and replace $x^T\alpha$ by λ every where.

$B = -A + \alpha\beta^T$; Replace $x^T\alpha$ by λ

$$P = \{(y, \pi_1) \mid A_i y - \pi_1 \leq 0, \forall i \in S_1; \quad y_j \geq 0, \quad \forall j \in S_2; \ \Sigma_{j \in S_2} y_j = 1\}$$

$$Q = \{(x, \pi_2) \mid \quad x_i \geq 0, \quad \forall i \in S_1; \ x^T B^j - \pi_2 \leq 0, \forall j \in S_2; \ \Sigma_{i \in S_1} x_i = 1\}$$

$$Q' = \{(x, \lambda, \pi_2) \mid \ x_i \geq 0, \ \forall i; \quad -x^T A^j + \lambda\beta_j - \pi_2 \leq 0, \ \forall j; \ \Sigma_{i \in S_1} x_i = 1\}$$

- $H_\alpha : \lambda - x^T\alpha = 0$. $Q = Q' \cap H_\alpha$.

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QP':

$$\max: \quad \lambda(\beta^T y) - \pi_1 - \pi_2$$
$$(y, \pi_1) \in P, \ (x, \lambda, \pi_2) \in Q'$$

In $P \times Q'$, $\lambda(\beta^T y) - \pi_1 - \pi_2 \le 0$. Equality iff fully-labeled.

Solutions of QP'

$$\lambda(\beta^T y) - \pi_1 - \pi_2 \leq 0.$$ Equality iff fully-labeled.

- $\mathcal{N}$: Solutions of QP'. Set of fully-labeled points of $P \times Q'$.
- NE of $(A, B) \leftrightarrow \mathcal{N} \cap H_\alpha$.
  - Recall: $Q = Q' \cap H_\alpha$. NE iff fully-labeled in $P \times Q$.

Goal: Find a point in $\mathcal{N} \cap H_\alpha$.

Structure of $\mathcal{N}$

Recall: Fully-labeled points of $P \times Q$ are vertex pairs.

In $P \times Q'$, λ gives an extra degree of freedom for fully-labeled.

- $\mathcal{N}$: one infinite path, and may be a set of cycles on 1-skeleton.
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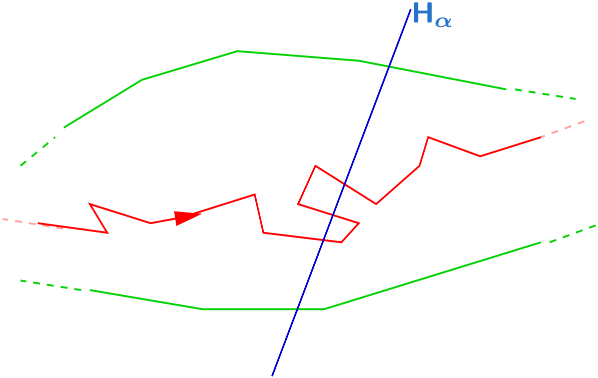
$$OPT(\delta) \text{ - } \quad \begin{array}{ll} \max: & \delta(\beta^T y) - \pi_1 - \pi_2 \\ & (x, \pi_1) \in P,\ (y, \lambda, \pi_2) \in Q', \lambda = \delta \end{array}$$

$$\mathcal{N}(a) \neq \emptyset, \text{ and } OPT(a) = \mathcal{N}(a),\ \forall a \in \mathbb{R}.$$

- **$\mathcal{N}$ forms a single path with λ being monotonic.**

Enumeration Algorithm

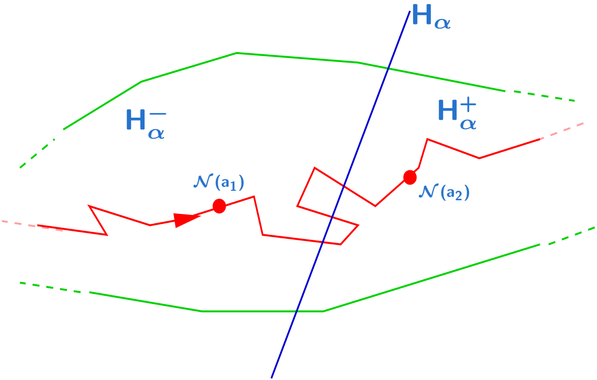
- Recall: NE of $(A, B) \leftrightarrow \mathcal{N} \cap H_\alpha$.



- Follow the path $\mathcal{N}$ and output whenever hit H_α (complementary pivot).

The Efficient Algorithm

- Recall: NE of $(A, B) \leftrightarrow \mathcal{N} \cap H_\alpha$. $H_\alpha : \lambda - x^T\alpha = 0$
- $a_1 = \min_i \alpha_i$, $a_2 = \max_i \alpha_i$; $a_1 \leq \sum_i \alpha_i x_i \leq a_2, \ \forall x \in \Delta_1$.



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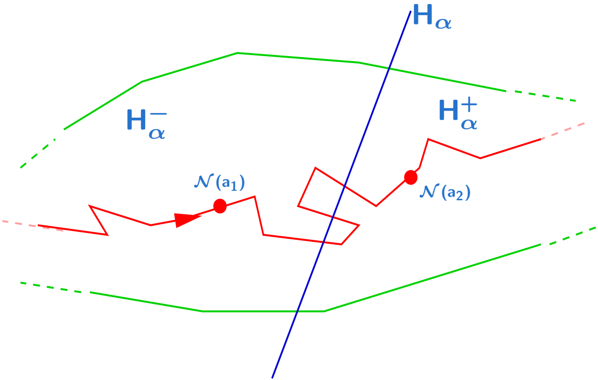

- Since $\mathcal{N}(a) = OPT(a)$, obtain it by solving an LP.

The Efficient Algorithm: BinSearch

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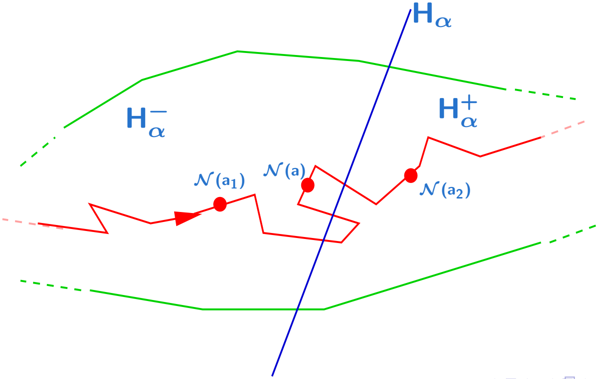
BinSearch:

1. Let $a = \frac{a_1+a_2}{2}$. If $\mathcal{N}(a) \in H_\alpha$ then output NE; Exit.



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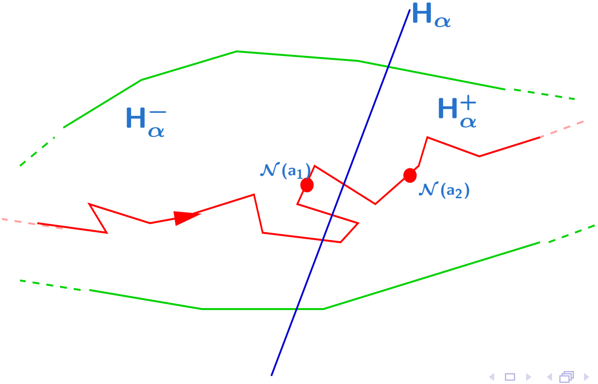

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Time Complexity (polynomial time):

- Solve an LP to obtain $\mathcal{N}(a)$ in Step 1.
- #iterations is $O(\mathcal{L})$, where $\mathcal{L}$ is the input bit length.

Homeomorphism

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- $P(\alpha) = P$ and $Q(\alpha) = Q' \cap H_\alpha$. NE of $G(\alpha) \leftrightarrow \mathcal{N} \cap H_\alpha$.

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- $P(\alpha) = P$ and $Q(\alpha) = Q' \cap H_\alpha$. NE of $G(\alpha) \leftrightarrow \mathcal{N} \cap H_\alpha$.

$\Gamma = \{G(\alpha) \mid \alpha \in \mathbb{R}^m\}$, and E_Γ be its NE correspondence.

- Γ is an m-dimensional subspace.
- $\mathcal{N}$ exactly covers E_Γ.
  - $(y, \pi_1), (x, \lambda, \pi_2) \in \mathcal{N} \Leftrightarrow (x, y)$ NE of $G(\gamma)$ with $\lambda = x^T\gamma$.
  - Proves $\mathcal{N}(a) \neq \emptyset$ and in turn $\mathcal{N}(a) = OPT(a)$.

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Extends to rank-k by considering $B = -A + \sum_{l=1}^{k} \alpha^l {\beta^l}^T$ and replacing $x^T \alpha^l$ by λ_l.

Replacing $-A$ by a Matrix C

Positives:

- Let $G(\alpha) = (A, \mathbf{C} + \alpha\beta^T)$.
- $\Gamma = \{G(\alpha) \mid \alpha \in \mathbb{R}^m\}$, and its NE correspondence E_Γ.
- The set $\mathcal{N}$ of fully-labeled points $\subset$ 1-skeleton of $P \times Q'$.
  - NE of $G(\alpha) \leftrightarrow \mathcal{N} \cap H_\alpha$. Exactly covers E_Γ.
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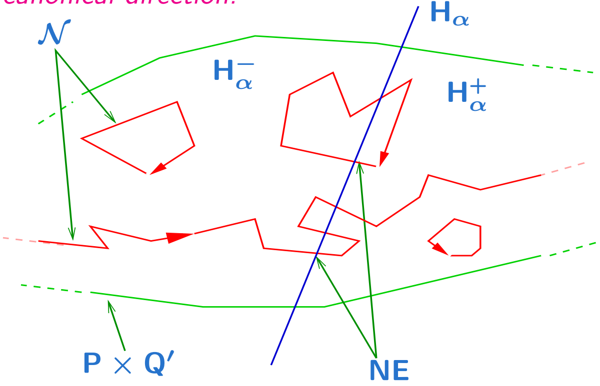
Negatives:

- $\mathcal{N}(a) \neq OPT(a)$.
  - $\mathcal{N}$ may indeed contain cycles.

Structure of $\mathcal{N}$

Proposition. *$\mathcal{N}$ consists of a set of cycles $\mathcal{C}_i$s and an infinite path $\mathcal{P}$, with a canonical direction.*



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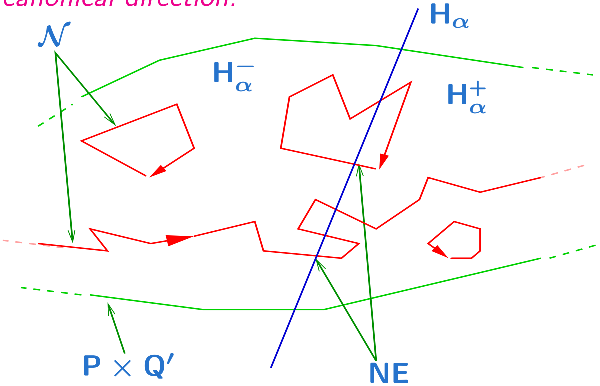

Consequences: Existence, Oddness and Index theorem.

Discussion

- What about rank-k? Are they hard?
  - The structural analysis for rank-k may help to answer.
- We resolved a special class of rank-1 QP; NP-hard in general.
  - Extend the technique to generalize this class.
- Structural: When does $\mathcal{N}$ contain only the path?

Thanks