Algebraic Circuit Complexity

Problem Set 1

Due date: September 3^{rd}, 2017

Instructions

1. The problem set has **4 questions** with a total score of **80 points**.
2. You are expected to work independently.
3. Solutions are expected as a LaTeX document. You may similar source files from the Algebra & Computation course on megh.
4. The deadline is 3rd September 2017. You can also submit answers to some (or all) of the questions **any time after the deadline** (a little before the course ends, of course; they need to be graded) for **half the credit**.

   This is to encourage you to solve all the question in these problem sets, even if it is past the deadline.

Questions

Question 1. *Assume that $f(x_1, \ldots, x_n)$ is computable by an* algebraic branching program (ABP) *of size s.*

1. **(10 points)** *Show that f is also computable by an algebraic circuit of size* $\text{poly}(s)$.

   What is the depth of the circuit that you constructed?

   That is, ABP $\subseteq$ Circuits

2. **(10 points)** *Additionally, if you knew that the ABP computing f is a* layered graph *(that is, the nodes are partitioned into disjoint layers and edges go only from layer i to layer $i+1$) with at most 10 vertices in each layer. Show that f is computable by an* algebraic formula *of size* $\text{poly}(s)$.

   That is (in general), $O(1)$-width-ABP $\subseteq$ Formulas.

Question 2. *We shall say that an ABP is* homogeneous *if*

Note that the underlying DAG is, in general, a multi-graph.

- *the nodes can be partitions into disjoint layers, with edges only going from layer i to layer $i+1$,*

- *all edge weights are of the form αx_i for some $\alpha \in \mathbb{F}$.*

(15 points) *Show that any ABP computing a degree-d homogeneous polynomial $f(x_1, \ldots, x_n)$ can be converted to a* homogeneous ABP *of size* $\text{poly}(s, d)$ *computing the same polynomial.*

Question 3. *Consider the* complete homogeneous symmetric polynomial *of degree d, denoted by $h_d(x_1, \ldots, x_n)$, that is a sum of all monomials of degree d (including non-multilinear monomials) with coefficient* 1 *each. For instance,*

$$h_3(x_1, x_2, \ldots, x_n) = \sum_i x_i^3 + \sum_{\substack{1 \le i,j \le n \\ i \ne j}} x_i^2 x_j + \sum_{1 \le i < j < k \le n} x_i x_j x_k.$$

(10 points) *For any d, show that the polynomial $h_d(x_1, \ldots, x_n)$ is computable by a* $\text{poly}(n, d)$*-size algebraic circuit over any large enough field.*

Question 4. *Say we are given an algebraic circuit $C(x_1, \ldots, x_n)$ of size s that computes a degree-d polynomial $f(x_1, \ldots, x_n)$.*

1. **(5 points)** *Show that the polynomial $\frac{\partial f}{\partial x_1}$ can be computed by an algebraic circuit without much blow-up in size, if $\mathbb{F}$ is large enough.*
2. **(5 points)** *What can you say if you are instead given a formula for f? Can you construct a formula computing $\frac{\partial f}{\partial x_1}$?*
3. **(5 points)** *What if we have a $\Sigma\Pi\Sigma$ circuit computing f? Can you construct a $\Sigma\Pi\Sigma$ circuit computing $\frac{\partial f}{\partial x_1}$ of not-much-larger size?*
4. **(5 points)** *What about computing $\frac{\partial^i f}{\partial x_1^i}$, for $i = \frac{d}{2}$ say?*
5. **(15 points)** *Suppose we want to compute the higher-order derivative*

$$g = \frac{\partial^{n/2} f}{\partial x_1 \, \partial x_2 \, \cdots \, \partial x_{n/2}}.$$

Given f as a small algebraic circuit, do you think it is possible to compute g by a small algebraic circuit? Try and justify your answer.