[CSS.324.1] Analysis of Boolean Functions (2020-II): Endsem

Due date: January 21st, 2021

Instructions

1. Solutions are to be submitted as a pdf via email. These could either be LaTeX-ed, or handwritten and scanned (you could do that using your mobile phone also), etc. When you email the pdf, please include the words tags "[CSS.324.1]" and "[Endsem]" in your email, and send me a private message on Zulip after you have emailed me.
2. You may find one of the questions useful for one of the later questions. Feel free to use the statements even if you haven't solved it.
3. The endsem has **10 questions** with a total score of **75 points**.

   The endsem is long only because of some exposition to give you the big picture and sufficient hints. Do not be alarmed :-)
4. This being an endsem, you SHOULD NOT discuss with other classmates
5. Please post any questions you may have on Zulip so that everyone has access to the clarifications.

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Almost the entirety of this endsem will be relating various complexity measures for Boolean Functions $f : \{T, F\}^n \to \{T, F\}$, some of which you already know what they are.

- $DT(f)$ will denote the depth of the smallest decision tree computing f.
- $\deg(f)$ will denote the degree of f, which is $\max \left\{ |S| \,:\, \hat{f}(S) \neq 0 \right\}$.
- $\text{sens}_x(f)$ is the number of coordinates of x that f is sensitive on. That is, $\text{sens}_x(f) = \left| \left\{ i \,:\, f(x^{\oplus i}) \neq f(x) \right\} \right|$. The sensitivity of f, denoted by $\text{sens}(f)$, is just the largest sensitivity over all inputs x. That is, $\text{sens}(f) = \max_x \text{sens}_x(f)$.

We will deal with some new measures of complexity and try to relate them to the above measures.

1 Certificate complexity and decision trees

Definition (Certificate complexity). *For a Boolean function $f : \{T, F\}^n \to \{T, F\}$ and an input x and let $f(x) = b$. A b-certificate for f at x intuitively is a sequence of bits in x when revealed "proves" that $f(x) = b$. Formally, a b-certificate for f at x is a partial assignment $w : S \to \{T, F\}$ such that*

- *(Consistency) for every $i \in S$, we have $w(i) = x_i$,*
- *(Sufficiency) $f(y) = b$ for any y that extends w.*

The b-certificate complexity of f, denoted by $C_b(f)$ is defined as

$$C_b(f) = \max_{x : f(x) = b} \; \min_{\substack{w : S \to \{T,F\} \text{ is a} \\ b\text{-certificate for } f \text{ at } x}} |S|$$

(the smallest number of bits to be revealed to prove that $f(x) = b$)

The certificate complexity of f, denoted by $C(f) = \max_{b \in \{T,F\}} C_b(f)$. ♢

Question 1. [2 points] Suppose $w : S \to \{T, F\}$ and $w' : S' \to \{T, F\}$ are respectively a T-certificate and a F-certificate respectively for a function f (on different inputs of course).

Show that $S \cap S' \neq \emptyset$, and in fact w and w' must be *inconsistent* on the intersection.

Question 2. [7 points] Suppose $f : \{T, F\}^n \to \{T, F\}$ be a function with T-certificate complexity k, then show that f can be computed by a DNF of width at most k.

Question 3. [7 points] Show that for any Boolean function $f : \{T, F\}^n \to \{T, F\}$, we have $C(f) \leq DT(f)$.

Question 4. In this question, you will see that $DT(f) \leq C(f)^2$. That is, we will build a decision tree for f that queries no more than $C(f)^2$ bits. Here is a rough sketch of how this will proceed. Complete the sketch and prove its correctness.

In phase i, let us assume that we have queried some sequence of bits so far. Let $f^{(i)}$ is the restricted function where the queried bits are frozen. If $f^{(i)}$ is the constant True function, then return True. Else, let $S_T(f^{(i)})$ and $S_F(f^{(i)})$ are all minimal T- and F-certificates for $f^{(i)}$ respectively. Pick an arbitrary F-certificate $w^{(i)} : S^{(i)} \to \{T, F\}$ and query x at the locations $S^{(i)}$. If x and $w^{(i)}$ agree on $S^{(i)}$, return False. Else, proceed to the next phase.

- **[5 points]** Show that every phase queries at most $C(f)$ bits of x (in fact, at most $C_F(f)$ bits).
- **[10 points]** Show that the number of phases is at most $C_T(f)$

  (Hint: What happens to $C_T(f^{(i)})$ as i increase? Question 1 might be useful here...)

2 Block-sensitivity

Definition 2.1 (Block sensitivity). *For a boolean function $f : \{T, F\}^n \to \{T, F\}$, and an input x, the* block sensitivity *of f at x, denoted by* b-sens$_x(f)$ *is the largest possible k such that there are disjoint subsets $S_1, \ldots, S_k \subseteq [n]$ such that $f(x) \neq f(x^{\oplus S_i})$ where $x^{\oplus S_i}$ denotes the input obtained from x by flipping all the bits in S_i.*

The block sensitivity *of f, denoted by* b-sens(f) *is defined as* $\max_x$ b-sens$_x(f)$.

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Question 5. [5 points] Show that sens$(f) \leq$ b-sens$(f) \leq C(f)$.

Question 6. In this question you will show that b-sens$(f) \geq \sqrt{C(f)}$.

Pick an input x that maximises the certificate complexity of f at x. Let S_1 be the smallest set of indices such that $f(x) \neq f(x^{\oplus S_1})$. Let S_2 then be the smallest set of indices disjoint from S_1 such that $f(x) \neq f(x^{\oplus S_2})$, and so on until we pick S_k beyond which no such set exists.

- **[3 points]** Show that $\bigcup S_i$ must be a certificate for f at x. Therefore, $\sum |S_i| \geq C(f)$.
- **[3 points]** By the minimality of each S_i, show that f on $x^{\oplus S_i}$ is sensitive to every bit of S_i. Therefore, b-sens$_{x^{\oplus S_i}}(f) \geq |S_i|$.
- **[4 points]** Deduce from the above two points that b-sens$(f) \geq \sqrt{C(f)}$ (Hint: Either $k \geq \sqrt{C(f)}$ or not...)

3 Degree and block sensitivity

You should recall the following simple fact from our lectures that $\deg(f) \leq DT(f)$. And what we have seen above is that $DT(f)$ is polynomially related to $C(f)$ which in-turn is polynomially related to b-sens(f). In this section, we'll relate things in the other direction. We'll need the following fact that you can assume without proof.

Fact 3.1. Let $g : \{T, F\}^n \to \{T, F\}$ be such that $g(F, \ldots, F) = F$ and $g(x) = T$ for all string x with exactly one T in it. Then, $\deg(g) \geq \sqrt{n/2}$.

Question 7. Suppose $f : \{T, F\}^n \to \{T, F\}$. Let x be any fixed input let $S_1, \ldots, S_k$ be disjoint subsets of $[n]$. Define

$$g(y_1, \ldots, y_k) = f(x^{\oplus y_1 S_1 \cup \ldots \cup y_k S_k}),$$

$$\text{where } y_i S_i = \begin{cases} S_i & \text{if } y_i = T \\ \emptyset & \text{otherwise.} \end{cases}$$

In words, $g(y_1, \ldots, y_k)$ is obtained by flipping x in the blocks that correspond to y_i's being T and applying f on the resulting string.

- **[5 points]** Show that $\deg(g) \leq \deg(f)$
- **[5 points]** By choosing x, $S_1, \ldots, S_k$ carefully, deduce that $\deg(g) \geq \sqrt{\text{b-sens}(f)/2}$.

You have now seen that $C(f)$, $DT(f)$, $\deg(f)$, b-sens(f) are all polynomially related to each other. The obvious question is what about sens(f)? The connections you have seen so far shows that $\text{sens}(f) \leq \text{poly}(\deg(f))$. But can we also lower bound sensitivity by a polynomial function of the degree?

4 The Sensitivity Conjecture

The following conjecture remained open for a long time, until it was resolved by Hao Huang [Hua19] very recently in a *really short and elegant* proof. But we'll build up the back story for this, and those interested (and you really should be interested!) can read the Huang's proof.

Conjecture 4.1 (Sensitivity Conjecture)**.** There is a constant $\alpha > 0$ such that for any boolean function f, we have $\text{sens}(f) \geq (\deg f)^\alpha$.

Question 8. [3 points] Show that Conjecture 4.1 is equivalent to proving it only for functions $f : \{-1, 1\}^n \to \{-1, 1\}$ of degree n.

(Hint: Suppose f has degree d, can you work with suitable restriction of f to just d variables and apply Conjecture 4.1 to that?)

Question 9. Suppose $f : \{-1,1\}^n \to \{-1,1\}$ be a Boolean function and let $g(x) = f(x) \cdot x_1 \cdots x_n$.

- **[3 points]** Show that $\text{sens}_x(f) = n - \text{sens}_x(g)$.
- **[3 points]** Show that if $\deg(f) = n$ then $\mathbb{E}_x[g(x)] \neq 0$

Question 10. [10 points] Show that the sensitivity conjecture is equivalent to the following statement about induced subgraphs of the boolean hypercube:

> Let V be a subset of vertices of the hypercube $\{1, -1\}^n$ with $|V| \neq 2^{n-1}$, and let $G(V)$ be the *induced subgraph* on the vertices V. Then, there is some vertex $x \in V$ whose degree* in G is at least n^α.

**: This degree is graph degree... number of neighbours*

(Hint: Interpret the set V as a boolean function $f : \{-1,1\}^n \to \{-1,1\}$ and consider g constructed in the previous question. Can you relate $\text{sens}_x(g)$ to the "degree" of the vertex x?)

Bonus: Suppose we remove the constraint that $|V| \neq 2^{n-1}$, can you see why you cannot give any meaningful lower bound on the maximum degree of the induced subgraph?

Remark. *Huang [Hua19] gave a proof of the above statement with* $\alpha = 0.5$*. That resolved the sensitivity conjecture by proving that* $\text{sens}(f) \geq \sqrt{\deg(f)}$. ◊

[Hua19] Hao Huang. Induced subgraphs of hypercubes and a proof of the sensitivity conjecture. *Annals of Mathematics*, 190(3):949–955, 2019. URL: https://www.jstor.org/stable/10.4007/annals.2019.190.3.6.