[CSS.324.1] Analysis of Boolean Functions (2020-II)

Problem Set 1

Due date: October 23rd, 2020

Instructions

1. Solutions are to be submitted as a pdf via email. These could either be LaTeX-ed, or handwritten and scanned (you could do that using your mobile phone also), etc. When you email the pdf, please include the words tags "[CSS.324.1]" and "[PS1]" in your email, and send me a private message on Zulip after you have emailed me.
2. All of these problems are essentially from Ryan O'Donnell's text [O'D14].
3. The problem set has **15 questions** with a total score of **100 points**.

   (There are lots of questions in this set so do not wait until very close to the deadline to start working on it! Try and do one question a day.)
4. You are welcome to discuss with other classmates **as long as these discussion are reasonable**; these are not meant for one to solve the problem for the other. You are eventually expected to find and write your own solutions.

   If you do discuss, you are expected to explicitly mention who you discussed with and which parts of your solution came from these discussions.
5. Please post any questions you may have on Zulip so that everyone has access to the clarifications.

Question 1. (no points for this but mandatory)

Calculate the Fourier representation for the following functions:

1. $\min_3 \{-1,1\}^3 \to \{-1,1\}$, that computes the min of the three bits.
2. The inner-product function $\mathrm{IP}_n : \{-1,1\}^{2n} \to \{-1,1\}$ given by

$$\mathrm{IP}_n(\mathbf{x},\mathbf{y}) = (-1)^{\langle \mathbf{x},\mathbf{y}\rangle} = (-1)^{\sum x_i y_i}$$

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3. $\text{Maj}_5 : \{-1,1\}^5 \to \{-1,1\}$

For each of the above functions, compute Inf_1 as well.

Question 2. [5 points] Many times, we will be working with functions that are more naturally defined as $f : \mathbb{F}_2^n \to \mathbb{R}$. For a vector $\mathbf{v} \in \mathbb{F}_2^n$, define the function $\tau_{\mathbf{v}} : \mathbb{F}_2^n \to \{-1,1\}$ given by

$$\tau_{\mathbf{v}}(\mathbf{x}) = (-1)^{\langle \mathbf{v}, \mathbf{x} \rangle} = (-1)^{\sum x_i v_i}.$$

1. Briefly argue that any function $f : \mathbb{F}_2^n \to \mathbb{R}$ can be uniquely written as a linear combination of the functions $\{\tau_{\mathbf{v}} \ : \ \mathbf{v} \in \mathbb{F}_2^n\}$.

2. Suppose $\tilde{f}(\mathbf{v})$ is the coefficient of $\tau_{\mathbf{v}}$ in the above linear combination for f. Consider the modified function F, with domain $\{-1,1\}^n$ given by

$$F : \{-1,1\}^n \to \mathbb{R}$$
$$F(x_1, \ldots, x_n) = f\left(\frac{1-x_1}{2}, \ldots, \frac{1-x_n}{2}\right).$$

   Then, $\tilde{f}_{\mathbf{v}} = \hat{F}(S)$, where $S = \{i \in [n] \ : \ \mathbf{v}_i = 1\}$.

(Basically, we shouldn't be too tied to the domain being $\{-1,1\}^n$; we can instead change our perspective on what the characters are; the Fourier coefficients remain the same in both cases. Hopefully this problem should make you more comfortable with the domain either being $\{-1,1\}^n$ or $\{0,1\}^n$.)

Question 3. [5 points] Recall that a subspace $A \subseteq \mathbb{F}_2^n$ is a subset of $\mathbb{F}_2^n$ which is also a vector space over $\mathbb{F}_2$.

Suppose $f_A : \mathbb{F}_2^n \to \{0,1\}$ is the indicator function of a subspace A; i.e., $f(\mathbf{v}) = 1$ if and only if $\mathbf{v} \in A$. Show that the Fourier representation of A is only supported on the *dual subspace* $A^{\perp}$ defined as

$$A^{\perp} := \left\{\mathbf{v} \in \mathbb{F}_2^n \ : \ \langle \mathbf{u}, \mathbf{v} \rangle = \sum u_i v_i \bmod 2 = 0 \text{ for all } \mathbf{u} \in A\right\}.$$

That is, the coefficient of $\chi_{\mathbf{v}}$ in the Fourier representation of f_A is nonzero if and only if $\mathbf{v} \in A^{\perp}$.

Also, can you also compute all the nonzero Fourier coefficients of f_A?

Question 4. [5 points] Let $f : \{-1,1\}^n \to \mathbb{R}$. Then, its Fourier representation $f(\mathbf{x}) = \sum_S \hat{f}(S)\mathbf{x}^S$ in fact gives an extension of $F : \mathbb{R}^n \to \mathbb{R}$ (nothing is stopping us from feeding arbitrary real vectors to the right hand side!)

Suppose $\mu = (\mu_1, \ldots, \mu_n) \in \{-1,1\}^n$, then show that

$$F(\mu) = \mathop{\mathbb{E}}_{\mathbf{y}}[f(\mathbf{y})]$$

where the expectation is over a distribution on $\{-1,1\}^n$ such that $\mathbb{E}[y_i] = \mu_i$, and the coordinates are independent of one another.

Question 5. [5 points] Suppose $f : \{-1,1\}^n \to \{-1,1\}$ and $f = f^{=1}$. Show that f must be $\pm\chi_{\{i\}}$ for some $i \in [n]$.

Question 6. [5 points] Prove that there are no functions $f : \{-1,1\}^n \to \{-1,1\}$ with exactly two nonzero Fourier coefficients.

Are there $\{-1,1\}$-valued functions with exactly three nonzero Fourier coefficients?

Question 7. [5 points] Let $f : \{-1,1\}^n \to \{-1,1\}$. A coordinate $i \in [n]$ is said to be b-pivotal for f at $\mathbf{x} \in \{-1,1\}^n$ if $f(\mathbf{x}) = b$ and $f(\mathbf{x}^{\oplus i}) = -b$. Deduce that, for any $b \in \{-1,1\}$,

$$\mathbb{I}(f) = 2 \cdot \mathop{\mathbb{E}}_{\mathbf{x}}[\text{number of } b\text{-pivotal coordinates for } f \text{ at } x].$$

Question 8. [5 points] If $f : \{-1,1\}^n \to \{-1,1\}$ is a monotone function, then show that

$$\mathbb{I}(f) \leq \Theta(\sqrt{n}).$$

Hint: Recall the argument about the number of voters agreeing with an election outcome.

Question 9. [5 points] Show that for any function $f : \{-1,1\}^n \to \{-1,1\}$, we have $\mathrm{NS}_\delta(f) \leq \delta \cdot \mathbb{I}(f)$.

Hint: You may want to use the fact that $(1-\delta)^r \geq 1 - \delta r$ whenever $\delta \leq 1$ and $r \in \mathbb{R} \setminus (0,1)$

Question 10. [10 points] Let $f : \mathbb{F}_2^n \to \{-1,1\}$. Suppose $\mathrm{dist}(f, \chi_{S^*}) = \delta$ for some $S^* \subseteq [n]$.

1. Show that for all $S \neq S^*$, we have $\left|\hat{f}(S)\right| \leq 2\delta$.
2. Deduce that the BLR test rejects f with probability at least $3\delta - 10\delta^2 + 8\delta^3$.
3. Show that this bound cannot be improved to $c\delta - O(\delta^2)$ for any constant $c > 3$.

Hint: You may want to see what $\mathrm{dist}(\chi_S, \chi_{S^*})$ *is*

Question 11. [10 points] A function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is said to be an *affine (linear) function* if there is an $\mathbf{a} \in \mathbb{F}_2^n$ and $b \in \mathbb{F}_2$ such that $f(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle + b = (\sum a_i x_i) + b \bmod 2$.

1. Show that f is affine if and only if $f(\mathbf{x}) + f(\mathbf{y}) + f(\mathbf{z}) = f(\mathbf{x} + \mathbf{y} + \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{F}_2^n$.
2. Modify the BLR test suitably so that you now have a property tester to check if a given blackbox is computing an affine function.

That is, construct the test that makes only constantly many queries to the blackbox and outputs "Yes" or "No" such that

- If f is affine, the tester *always* says "Yes"
- If $\mathrm{dist}(f, \mathrm{affine}) = \varepsilon$, then the tester says "No" with probability at least $1 - O(\varepsilon)$.

Question 12. [10 points] Let $n = w \cdot 2^w$. Consider the function $f = \mathrm{Tribes}_{w,2^w} : \{-1,1\}^n \to \{-1,1\}$ (recall $\mathrm{Tribes}_{w,s} = \mathrm{OR}_s(\mathrm{AND}_w, \ldots, \mathrm{AND}_w)$).

1. Calculate $\mathbb{E}[f]$ and $\mathrm{Var}(f)$, asymptotically as a function of n.
2. Describe the function $D_1(f)$.
3. Compute $\mathrm{Inf}_1(f)$ and $\mathbb{I}(f)$, asymptotically as a function of n.

Question 13. [10 points] Calculate $\mathrm{Stab}_\rho(\mathrm{Tribes}_{w,s})$ by using the following (or otherwise).

Slightly painful calculation, but worth it.

1. Suppose $(\mathbf{x}, \mathbf{y}) \in \{-1,1\}^{2w}$ is a ρ-correlated pair, calculate the probability the event "$\mathrm{AND}_w(\mathbf{x}) = 1$ and $\mathrm{AND}_w(\mathbf{y}) = 1$ (i.e., both ANDs are false)".
2. For $(\mathbf{x}, \mathbf{y}) \in \{-1,1\}^n \times \{-1,1\}^n$ that is a ρ-correlated pair, calculate the probability of the event "$\mathrm{Tribes}_{w,s}(\mathbf{x}) = 1$ and $\mathrm{Tribes}_{w,s}(\mathbf{y}) \neq -1$" based on what happens at each individual AND_w gate.
3. Use these to calculate $\mathrm{Stab}_\rho(\mathrm{Tribes}_{w,s})$.

It might be easier to write some of these expressions in terms of $\delta = \frac{1-\rho}{2}$.

Question 14. [10 points] Arrow's theorem is usually stated in the more generate setting where the aggregate functions for the three head-to-head elections (for the 3-candidate setting) need not be the same function. Turns out, this doesn't change anything:

Suppose f, g, h are the three *unanimous* aggregate functions for the three head-to-head elections and suppose they *always* give rise to a Condorcet winner. Show that $f = g = h$.

(Hint: Try and find relations between, say f and g, by trying out voting scenarios such with voter i casting votes $(x_i, -x_i, f(\mathbf{x}))$ etc.)

Question 15. [10 points] Pick up any other question from the first 2 chapters of the text [O'D14], give your solution, and also say why you chose this question.

The scoring is not just for the solution, but also for your choice of the question.

[O'D14] Ryan O'Donnell. *Analysis of Boolean Functions*. Cambridge University Press, 2014. Available from http://www.contrib.andrew.cmu.edu/~ryanod/?page_id=2334.