[CSS.324.1] Analysis of Boolean Functions (2020-II)

Problem Set 2

Due date: November 20^{th}, 2020

Instructions

1. Solutions are to be submitted as a pdf via email. These could either be LaTeX-ed, or handwritten and scanned (you could do that using your mobile phone also), etc. When you email the pdf, please include the words tags "[CSS.324.1]" and "[PS2]" in your email, and send me a private message on Zulip after you have emailed me.
2. All of these problems are essentially from Ryan O'Donnell's text [O'D14].
3. You may find one of the questions useful for one of the later questions. Feel free to use the statements even if you haven't solved it.
4. The problem set has **7 questions** with a total score of **80 points**.

   (Again, do not wait until very close to the deadline to start working on it! Try and do one question a day.)
5. You are welcome to discuss with other classmates **as long as these discussion are reasonable**; these are not meant for one to solve the problem for the other. You are eventually expected to find and write your own solutions.

   Actually, given this weird COVID era, I would strongly encourage you to discuss with your classmates.

   If you do discuss, you are expected to explicitly mention who you discussed with and which parts of your solution came from these discussions.
6. Please post any questions you may have on Zulip so that everyone has access to the clarifications.

Question 1. [10 points]

1. A function $f : \{-1,1\}^n \to \mathbb{R}$ is said to be *even* if $f(-\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x}$, and *odd* if $f(-\mathbf{x}) = -f(\mathbf{x})$ for all $\mathbf{x}$.

   Show that, for even/odd functions, its Fourier coefficients are nonzero only for sets of even/odd sized sets respectively.

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2. If f is an *even* function, prove the strengthening of the the Poincare inequality:

$$\mathrm{Var}(f) \leq \frac{1}{2} \cdot \mathbb{I}(f).$$

3. For a function $f : \{-1,1\}^n \to \mathbb{R}$, define its *dual* $f^\dagger : \{-1,1\}^n \to \mathbb{R}$ as

$$f^\dagger(\mathbf{x}) = -f(-\mathbf{x}).$$

Describe the Fourier coefficients of $f^\dagger$ in terms of those of f.

If f is a CNF, what can you say about $f^\dagger$?

Question 2. [10 points] If $f : \{-1,1\}^n \to \{-1,1\}$ that is computable by a depth-d, layered circuit of size s and width w. Use the Switching Lemma to show that $\mathbb{I}(f) = O\left(w(\log s)^{d-2}\right)$ (Hint: You may want to use induction on d.)

Use this to deduce that any function $f : \{-1,1\}^n \to \{-1,1\}$ that is computable by a depth-d, size s, layered circuit must have $\mathbb{I}(f) \leq O((\log s)^{d-1})$.

Question 3. [10 points] Similar to the D_i operator that we discussed in class, there is a related operator called the *Laplace* operator $L(\cdot)$ given by

$$Lf(\mathbf{x}) = \sum_i (L_i f)(\mathbf{x}),$$

$$\text{where} \quad (L_i f)(\mathbf{x}) = \frac{f(x) - f(x^{\oplus i})}{2}.$$

1. Compute the Fourier representation of Lf in terms of the Fourier representation of f.
2. Show that $\langle f, Lf \rangle = \mathbb{I}(f)$.

Question 4. [10 points] For a function $f : \{-1,1\}^n \to \mathbb{R}$. Assume that $\mathbb{E}[|f(x)|] = 1$.

1. Suppose $\mathcal{F} = \left\{S \ : \ \hat{f}(S) \neq 0\right\}$, show that $\|\hat{f}\|_2^2 = \sum_S \hat{f}(S)^2 \leq |\mathcal{F}|$.
2. Suppose $\mathcal{G} = \{\mathbf{x} \ : \ f(\mathbf{x}) \neq 0\}$, show that $\|f\|_2^2 = \mathbb{E}_x[f(x)^2] \geq 2^n / |\mathcal{G}|$.
3. Deduce the Uncertainity principle:

   Let $f : \{-1,1\}^n \to \mathbb{R}$ be a nonzero function. Then,

   $$\left|\left\{S \subseteq [n] \ : \ \hat{f}(S) \neq 0\right\}\right| \cdot \left|\{\mathbf{x} \in \{-1,1\}^n \ : \ f(\mathbf{x}) \neq 0\}\right| \geq 2^n$$

   In other words, if f is "sparse" on the truthtable, it cannot be "sparse" in the Fourier representation; and vice-versa.

The statement isn't insisting that $\mathbb{E}[|f(x)|] \neq 1$.

Question 5. [15 points] Let $\mathbf{w}_1, \ldots, \mathbf{w}_n \in \mathbb{R}^d$ be vectors and let $\|\mathbf{v}\| = \sqrt{\sum v_i^2}$, the usual Euclidian norm. Given this set of vectors, define the function $g : \{-1,1\}^n \to \mathbb{R}$ given by $g(\mathbf{x}) = \|x_1\mathbf{w}_1 + \ldots + x_n\mathbf{w}_n\|$.

1. Show that, for any $\mathbf{x} \in \{-1,1\}^n$, we have $(Lg)(\mathbf{x}) \leq g(\mathbf{x})$. (Hint: Use the triangle inequality.)
2. Use this to show that $2\,\mathrm{Var}(g) \leq \|g\|^2 = \langle g, g\rangle$.
3. Deduce the Khintchine-Kahane inequality that states that

$$\mathop{\mathbb{E}}_{\mathbf{x}}\left[\left\|\sum_i x_i \mathbf{w}_i\right\|\right] \geq \frac{1}{\sqrt{2}} \cdot \mathop{\mathbb{E}}_{\mathbf{x}}\left[\left\|\sum_i x_i \mathbf{w}_i\right\|^2\right]^{1/2}.$$

4. Show that the constant $\sqrt{2}$ is tight ($d = 1$ should be sufficient to find an example).

Now that you know the Khintchine-Kahane inequality, you can read Theorem 5.2 in the text that proves that $W^{\leq 1}(\mathrm{LTF}) \geq \frac{1}{2}$.

Question 6. [15 points] Solve one of the following two questions presented as Option 1 and Option 2. (You get **[5 bonus points]** if you can convince one of your classmates to solve the question other than the one you solved.)

Option 1: A proof of the *Baby Switching Lemma* (with a slightly different constant).

> **Lemma.** Let f be computable by a DNF of width w. Then, for all $\delta \geq 0$,
>
> $$\Pr_{(J|\mathbf{z})\sim\delta \text{ r.r.}}\left[f_{J|\mathbf{z}} \text{ is not constant}\right] \leq 3\delta w.$$

Let's assume $f = T_1 \vee T_2 \vee \ldots \vee T_r$. We may assume that $\delta \leq 1/3$, otherwise there is nothing to prove.

1. Let $R = J \mid \mathbf{z}$ be a "bad" restriction in the sense that $f_{J|\mathbf{z}}$ is not constant. Let i be the smallest index such that T_i under the restriction R is not constant. And let j be the first index such that either x_j or $\overline{x_j}$ is a live literal in T_i restricted by R. Show that there is a unique restriction $R' = (J \setminus \{j\}) \mid \mathbf{z}'$ that *extends* R (by additionally assigning a value for x_j) that *does not* falsify T_i.
2. Imagine the scenario of running through each "bad" restriction R and writing down the unique restriction R' defined by the process above. Show that each R' is written down at most w times.

(In other words, given the *extended restriction* R', there are at most w candidate "bad" restrictions that would result in this R' when following the process described above).

3. If R' is obtained from R are defined as above, then

$$\Pr_{\delta \text{ r.r.}}[R] = \left(\frac{2\delta}{1-\delta}\right) \cdot \Pr_{\delta \text{ r.r.}}[R'] .$$

4. Deduce that $\Pr[f_{J|\mathbf{z}} \text{ is not constant}] \leq 3\delta w$.

OPTION 2: Tighter bounds on the influence of small-width DNFs.

Let f be a function computed by a CNF (which is an AND of ORs) of width w.

Consider the following procedure to try and find an $\mathbf{x}$ such thaf $f(\mathbf{x}) = -1 = \text{True}$. Choose a random permutation π of n elements. For each $i = 1, 2, \ldots, n$, check if $x_{\pi(i)}$ or $\overline{x_{\pi(i)}}$ is one of the clauses (the OR gates). If $x_{\pi(i)}$ is a clause *and* $\overline{x_{\pi(i)}}$ is also a clause, then return $\perp$ (failure).

If exactly one of $x_{\pi(i)}$ or $\overline{x_{\pi(i)}}$ is a clause, set the value of $x_{\pi(i)}$ it so that the f isn't falsified; we will say that x_i was *forced*. Simplify the clauses (remove clauses where one of the literals were set to True entirely, and throw away False literals from clauses).

Else, set x_i randomly to either 1 or -1.

1. Show that if a run of the above procedure did not return $\perp$, then the chosen $\mathbf{x}$ satisfies $f(\mathbf{x}) = \text{True}$.

2. For any $\mathbf{x} \in f^{-1}(\text{True})$, let $p(\mathbf{x})$ refer to the probability that the above procedure returns $\mathbf{x}$. Let I_j be the indicator random variable that the variable x_j was *forced*. Prove that

$$p(\mathbf{x}) = \mathbb{E}\left[\prod_{i=1}^{n}\left(\frac{1}{2}\right)^{1-I_j}\right].$$

3. Show that $p(\mathbf{x}) \geq \frac{1}{2^n} \cdot 2^{\left(\sum_j \mathbb{E}[I_j]\right)}$. (Hint: If the procedure did not return $\perp$, can it be that all the I_j's are zero?)

4. If $\mathbf{x}$ such that $f(\mathbf{x}) = \text{True}$ and $f(x^{\oplus j}) = \text{False}$, show that

$$\mathbb{E}[I_j \mid \text{procedure returned } \mathbf{x}] \geq \frac{1}{w},$$

where, recall, w is the width of the CNF computing f.

5. Deduce that $\mathbb{I}(f) \leq w$.

Question 7. [10 points] Pick up any other question from Chapters 3, 4 of the text [O'D14], give your solution, and also say why you chose this question.

The scoring is not just for the solution, but also for your choice of the question.

[O'D14] Ryan O'Donnell. *Analysis of Boolean Functions*. Cambridge University Press, 2014. Available from http://www.contrib.andrew.cmu.edu/~ryanod/?page_id=2334.