[CSS.324.1] Analysis of Boolean Functions (2020-II)

Problem Set 3

Due date: December 20^{th}, 2020

Instructions

1. Solutions are to be submitted as a pdf via email. These could either be LaTeX-ed, or handwritten and scanned (you could do that using your mobile phone also), etc. When you email the pdf, please include the words tags "[CSS.324.1]" and "[PS3]" in your email, and send me a private message on Zulip after you have emailed me.
2. All of these problems are essentially from Ryan O'Donnell's text [O'D14].
3. You may find one of the questions useful for one of the later questions. Feel free to use the statements even if you haven't solved it.
4. The problem set has **8 questions** with a total score of **75 points**.

   (Again, do not wait until very close to the deadline to start working on it! Try and do one question a day.)
5. You are welcome to discuss with other classmates **as long as these discussion are reasonable**; these are not meant for one to solve the problem for the other. You are eventually expected to find and write your own solutions.

   Actually, given this weird COVID era, I would strongly encourage you to discuss with your classmates.

   If you do discuss, you are expected to explicitly mention who you discussed with and which parts of your solution came from these discussions.
6. Please post any questions you may have on Zulip so that everyone has access to the clarifications.

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Question 1. Prove the following claims that were left as an exercise the lectures:

1. **[3 points]** For a $f \in L^2(\Omega^{\otimes n}, \pi^{\otimes n})$, show that

$$\mathrm{Inf}_i(f) = \mathop{\mathbb{E}}_{\mathbf{x}\sim\pi^{\otimes n}}\left[\mathop{\mathrm{Var}}_{x_i'\sim\pi}\left(f(x_1,\ldots,x_i',\ldots,x_n)\right)\right]$$

2. **[3 points]** In the p-biased world, show that

$$\mathbb{I}(f^{(p)}) = \mathop{\mathbb{E}}_{\mathbf{x}\sim\mu_p}\left[\mathrm{sens}_f(\mathbf{x})\right].$$

3. **[3 points]** Show that

$$f^{\subseteq J}(\mathbf{x}) = \sum_{\alpha:\ \alpha_i=0\ \ \forall i\in\overline{J}} \hat{f}(\alpha)\cdot\varphi_\alpha(\mathbf{x})$$

4. **[3 points]** Let $1 \le p$ and let p' be its Hölder dual. Show that, for any $g : \Omega \to \mathbb{R}$, there is an $f : \Omega \to \mathbb{R}$ with $\|f\|_p = 1$ such that $\|g\|_{p'} = \langle f, g\rangle$.

5. **[5 points]** For any real-valued random variable X with finite support Ω, let $\lambda = \min_{x\in\Omega}\Pr[X = x]$. Show that, for any $q \ge 2$,

$$\|X\|_q \le \left(\frac{1}{\lambda}\right)^{\frac{1}{2}-\frac{1}{q}}\|X\|_2.$$

Question 2 (Paley-Zygmund inequality **[3 points]**)**.** Let Z be a non-negative real random variable. Consider the following obvious fact:

$$\mathbb{E}[Z] = \mathbb{E}[Z\cdot\mathbb{1}_{Z\le\theta\mu}] + \mathbb{E}[Z\cdot\mathbb{1}_{Z>\theta\mu}]$$

where $\mu = \mathbb{E}[Z]$. Use this to prove that

$$\Pr[Z \ge \theta\mu] \ge (1-\theta)^2\cdot\frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]}.$$

(Pay attention to where you use the non-negativity of Z.)

Question 3. [5 points] Show that the number of (ε,δ)-notable coordinates for any $f : \{-1,1\}^n \to [-1,1]$ is $O\left(\frac{1}{\varepsilon\delta}\right)$. That is,

$$\left|\left\{i\in[n]\ :\ \mathrm{Inf}_i^{(1-\delta)}(f) > \varepsilon\right\}\right| \le \frac{C}{\varepsilon\delta}$$

for some absolute constant C.

Question 4. [10 points] Build two different Fourier bases for $L^2(\Omega, \text{uniform})$ where $\Omega = \{1, 2, 3\}$.

No cheating by replacing some ϕ with $-\phi$ and nonsense like that!

Write down the Fourier expansion of $f : \Omega^2 \to \mathbb{R}$ given by $f(x, y) = \mathbb{1}_{x=y}$ with respect to both the chosen bases. What is its total Influence?

Question 5. [15 points] A function $f \in L^2(\Omega^{\otimes n}, \pi^{\otimes n})$ is said to have "degree k" if f can be written as a sum of k-juntas. We will say $f : \Omega^{\otimes n} \to \{-1, 1\}$ is a *linear threshold function* if $f = \text{sign}(\ell(x))$ for a degree 1 function ℓ.

1. For strings $\omega^{(+1)}, \omega^{(-1)} \in \Omega^{\otimes n}$ and $\mathbf{x} \in \{-1, 1\}^n$, define the string $\omega^{(\mathbf{x})}$ as the string $(\omega_1^{(x_1)}, \ldots, \omega_n^{(x_n)}) \in \Omega^{\otimes n}$.

   Suppose $\omega^{(+1)}, \omega^{(+1)}$ are drawn independently according to $\pi^{\otimes n}$ and $\mathbf{x}, \mathbf{y}$ are ρ-correlated strings in $\{-1, 1\}^n$, show that $\omega^{(\mathbf{x})}, \omega^{(\mathbf{y})}$ is a ρ-correlated pair under $\pi^{\otimes n}$.

2. For $f \in L^2(\Omega^{\otimes n}, \pi^{\otimes n})$, and a pair $\omega^{(+1)}, \omega^{(-1)}$, define the function $g : \{-1, 1\}^n \to \mathbb{R}$ as

   $$g(\mathbf{x}) = f(\omega^{(\mathbf{x})}).$$

   Show that if f is a "linear threshold function", then $g : \{-1, 1\}^n \to \{-1, 1\}$ is a linear threshold function in the usual sense.

3. Show that a suitable analogue of Peres' theorem is true for "linear threshold functions" in $L^2(\Omega^{\otimes n}, \pi^{\otimes n})$. (You will have to first define your analogue of *noise sensitivity* in this setting)

Question 6. [5 points] Show that in order to prove a statement of the form $\|T_\rho f\|_q \le \|f\|_p$ for all f, it suffices to prove it just for non-negative functions.

(Hint: What happens when you replace f with $|f|$?)

Question 7. [5 points] Show that if X is (p, q, ρ)-hypercontractive for any $\rho < 1$, then $\mathbb{E}[X] = 0$.

Question 8. [15 points] In this question, you will prove that the Log-Sobolev inequality implies the Poincaré inequality.

1. For any $f : \{-1, 1\}^n \to \mathbb{R}$, show that $\text{Ent}((1 + \varepsilon f)^2) = 2\varepsilon^2 \text{Var}(f) + O(\varepsilon^3)$.

The Taylor expansion of $\varphi(x) = x \log x$ around 1 might come in handy.

2. Deduce the Poincaré inequality ($\text{Var}(f) \le \mathbb{I}(f)$ for all f) from the Log-Sobolev inequality ($\text{Ent}(f^2) \le 2 \cdot \mathbb{I}(f)$ for all f).

[O'D14] Ryan O'Donnell. *Analysis of Boolean Functions*. Cambridge University Press, 2014. Available from http://www.contrib.andrew.cmu.edu/~ryanod/?page_id=2334.