Derandomization from algebraic hardness

Treading the borders

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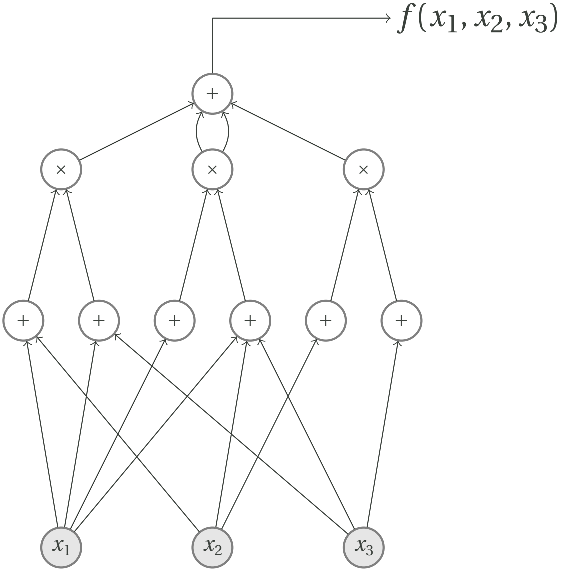
Algebraic Circuits



Aren't we all tired of this picture?

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These two problems are intimately connected to each other.

A "trivial" hitting set

Lemma ([Ore, Demillo-Lipton, Schwartz, Zippel])

If $P(x_1, \ldots, x_n)$ is a nonzero polynomial of degree d, and $S \subseteq \mathbb{F}$ of size at least $d+1$, then $P(\mathbf{a}) \neq 0$ for some $\mathbf{a} \in S^n$.

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A: Umm...

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- You care a lot about hard polynomials.
- Almost every polynomial is a hard polynomial.
- Find a hard polynomial.
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- You care a lot about hitting sets.
- Almost every set of poly-size is a hitting set.
- Find a hitting set.
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- You care a lot about hay.
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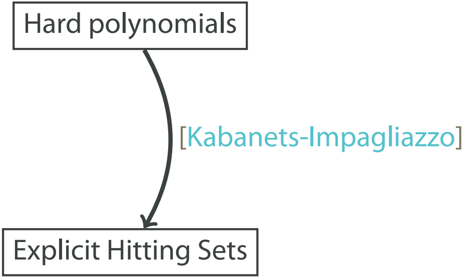
Question: Can we use one pseudorandom object to build another?

Lower bounds and hitting sets

Hard polynomials

Explicit Hitting Sets

Lower bounds and hitting sets



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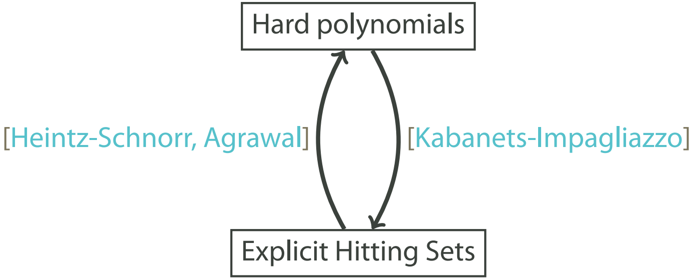
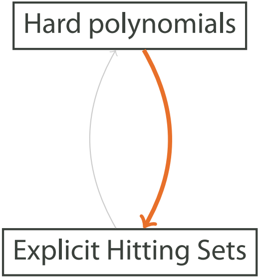


Lower bounds → hitting sets



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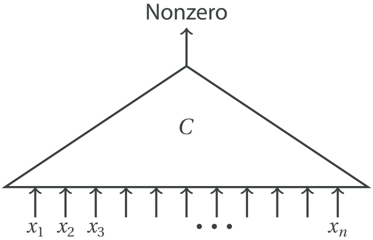

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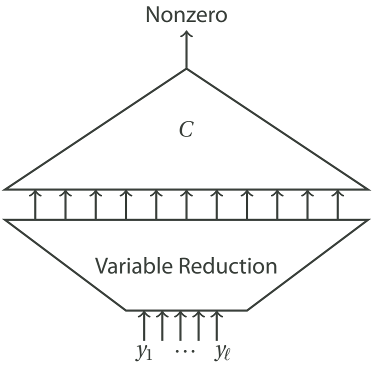

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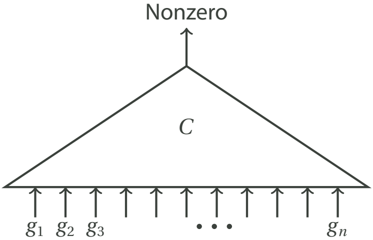

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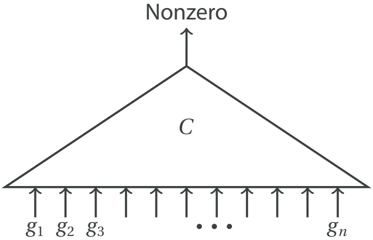

Definition (Generator)

A map $\mathcal{G} = (g_1, \ldots, g_n) \in \mathbb{F}[y_1, \ldots, y_\ell]^n$ is a hitting-set generator for a class $\mathscr{C}$ if

$$\forall\, C \in \mathscr{C} \quad , \quad C \neq 0 \Longleftrightarrow C \circ \mathcal{G} \neq 0.$$

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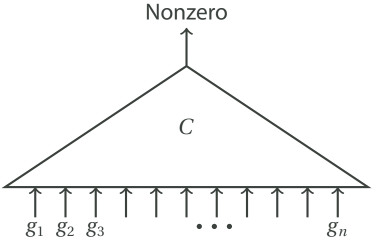

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Lemma

Let $\mathcal{G} = (g_1, \ldots, g_n) \in \mathbb{F}[y_1, \ldots, y_\ell]^n$ be an explicit hitting-set generator for $\mathcal{C}(n, D, s)$ of degree d. Then, we have

- *An explicit hitting set H of size $(dD+1)^\ell$*

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Our results

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Some consequences

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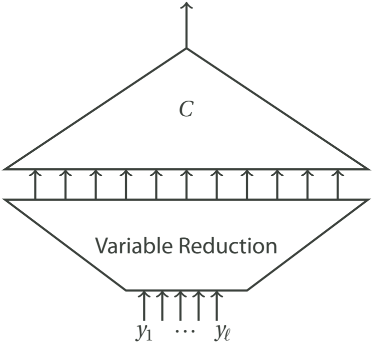
Hence $C \circ \mathscr{G}_P$ is a nonzero $2k$-variate polynomial of degree at most ds. Hence, we have a hitting set of size $(ds)^{2k} = s^{O(k^2/\varepsilon)}$. □

Revisiting variable reductions



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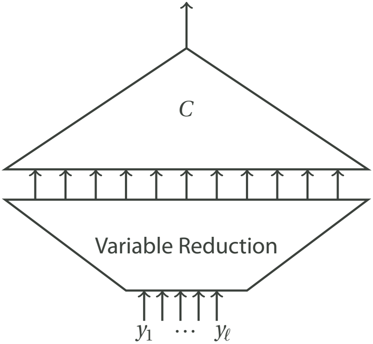

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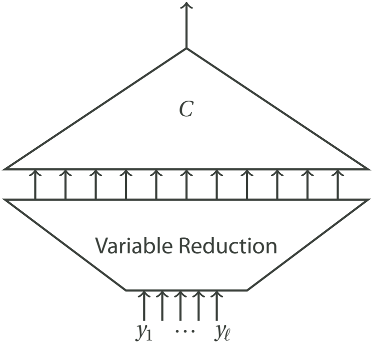

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Dream: $\text{size}(C \circ \mathscr{G}) \approx \text{size}(C) + \text{size}(\mathscr{G})$

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If P is a n-variate multilinear polynomial, then
$P \circ \mathscr{K}$ is a univariate polynomial of degree at most 2^n.

The Kronecker Map

$$\mathcal{K}_t = \left(1, y_1, y_1^2, \ldots, y_1^{2^{m-1}}, \ldots, 1, y_t, \ldots, y_t^{2^{m-1}}\right) \quad (n = t\,m)$$

$$x_1^{e_1} \cdots x_n^{e_n} \quad \longmapsto \quad y_1^{[e_1 \cdots e_m]_2} \cdots y_t^{[e_* \cdots e_n]_2}$$

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Consequences for bootstrapping

Theorem. [**Kumar-S-Tengse**]
Let $\varepsilon > 0$ and k (large enough) be fixed constants.

If, for all $s \geq k$, we have explicit hitting sets for $\mathscr{C}(k, s, s)$ of size

$$s^{k-\varepsilon},$$

then, we have explicit hitting sets for $\mathscr{C}(s, s, s)$ of size

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Consequences for bootstrapping

Corollary

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$$s^{O(1)}$$

Circuits and border are crucial for this.

What’s all this border stuff?

The Border

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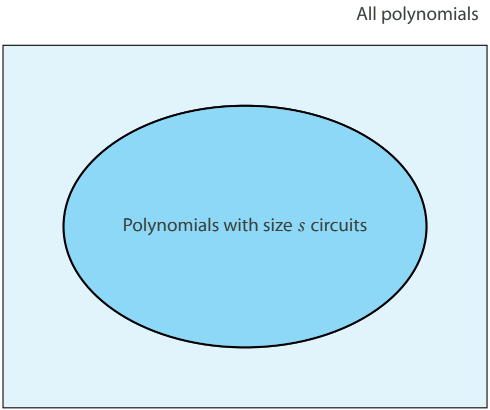

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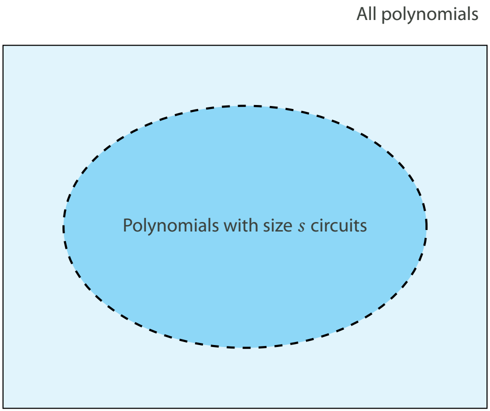

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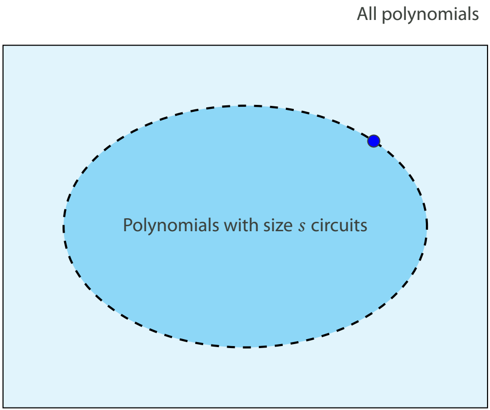

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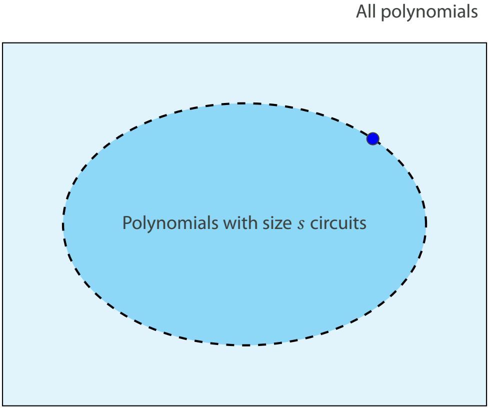

- Does not have size s circuits, but arbitrarily close to those that do.

Border computation: an example

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Hence, $x^{d-1}y \in \overline{\mathscr{C}}$ but not in $\mathscr{C}$.

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P_d can be $\overline{\text{computed}}$ in size s as well!

\begin{proof}

Designing generators

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technology

is indistinguishable

from magic

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Any sufficiently hard polynomial's evaluations

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[KI, NW] : $\mathcal{G} : (y_1, \ldots, y_\ell) \mapsto \left(P(\mathbf{y}|_{S_1}), \ldots, P(\mathbf{y}|_{S_n})\right)$

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'Taylored' appropriately

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Description of our generator

$$P(z_1, \ldots, z_k)$$

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$$P(\mathbf{y}+\mathbf{z}) = P(\mathbf{z}) + \sum_{i} y_i \cdot (\partial_i P)(\mathbf{z}) + \sum_{i,j} y_i y_j \cdot (\partial_{i,j} P)(\mathbf{z}) + \cdots$$

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Definition (The generator)

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The generator $\mathscr{G}_P$ is defined as

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Idea: *Think of* $C(\Delta_0(P), \ldots, \Delta_n(P)) = 0$ *as a* *differential equation* *and solve for* P.

Cauchy-Kowalevski Equations

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- Compute the homogeneous parts of P, one by one, via Newton Iteration

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(Assuming that $\mathscr{G}_P$ is *not* a generator)

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If $\tilde{C}(x_n) = C(g_0, \ldots, g_{n-1}, x_n) = 0$

$$C(x_0, \ldots, x_{n-1}, a) \neq 0$$
$$C(g_0, \ldots, g_{n-1}, a) = 0$$

Setting-up the initial conditions

(Assuming that $\mathscr{G}_P$ is *not* a generator)

Goal: Find a circuit C' of small size such that

$$
\begin{aligned}
C' \circ \mathscr{G}_P &= 0 \\
(\partial_n C') \circ \mathscr{G}_P &\neq 0.
\end{aligned}
$$

If $\tilde{C}(x_n) = C(g_0, \ldots, g_{n-1}, x_n) = 0$

$$
\begin{aligned}
C(x_0, \ldots, x_{n-1}, a) &\neq 0 \\
C(g_0, \ldots, g_{n-1}, a) &= 0
\end{aligned}
$$

Contradicts minimality!

Setting-up the initial conditions

(Assuming that $\mathscr{G}_P$ is *not* a generator)

Goal: Find a circuit C' of small size such that

$$C' \circ \mathscr{G}_P = 0$$
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$C(g_0, \ldots, g_{n-1}, g_n) = 0$

$(x_n - g_n)$ divides $\tilde{C}$

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(Assuming that $\mathscr{G}_P$ is *not* a generator)

Goal: Find a circuit C' of small size such that

$$\begin{aligned} C' \circ \mathscr{G}_P &= 0 \\ (\partial_n C') \circ \mathscr{G}_P &\neq 0. \end{aligned}$$

If $\tilde{C}(x_n) = C(g_0, \ldots, g_{n-1}, x_n) \neq 0$

$$\begin{aligned} C(g_0, \ldots, g_{n-1}, g_n) &= 0 \\ (\partial_n C)(g_0, \ldots, g_n) &= 0 \end{aligned}$$

$(x_n - g_n)^2$ divides $\tilde{C}$

Setting-up the initial conditions

(Assuming that $\mathscr{G}_P$ is *not* a generator)

Goal: Find a circuit C' of small size such that

$$
\begin{aligned}
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$$

If $\tilde{C}(x_n) = C(g_0, \ldots, g_{n-1}, x_n) \neq 0$

$$
\begin{aligned}
C(g_0, \ldots, g_{n-1}, g_n) &= 0 \\
(\partial_n C)(g_0, \ldots, g_n) &= 0 \\
(\partial_n^2 C)(g_0, \ldots, g_n) &= 0
\end{aligned}
$$

$(x_n - g_n)^3$ divides $\tilde{C}$

Setting-up the initial conditions

(Assuming that $\mathscr{G}_P$ is *not* a generator)

Goal: Find a circuit C' of small size such that

$$C' \circ \mathscr{G}_P = 0$$
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If $\tilde{C}(x_n) = C(g_0, \ldots, g_{n-1}, x_n) \neq 0$

$$C(g_0, \ldots, g_{n-1}, g_n) = 0$$
$$(\partial_n C)(g_0, \ldots, g_n) = 0$$
$$(\partial_n^2 C)(g_0, \ldots, g_n) = 0$$
$$\vdots$$
$$(\partial_n^r C)(g_0, \ldots, g_n) = 0$$

$(x_n - g_n)^{r+1}$ divides $\tilde{C}$

Setting-up the initial conditions

(Assuming that $\mathscr{G}_P$ is *not* a generator)

Goal: Find a circuit C' of small size such that

$$C' \circ \mathscr{G}_P = 0$$
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$$(\partial_n C)(g_0, \ldots, g_n) = 0$$
$$(\partial_n^2 C)(g_0, \ldots, g_n) = 0$$
$$\vdots$$
$$(\partial_n^r C)(g_0, \ldots, g_n) = 0$$

$(x_n - g_n)^{r+1}$ divides $\tilde{C}$

$(x_n - g_n)^t$ cannot divide $\tilde{C}$
if $t > \deg \tilde{C}$

Setting-up the initial conditions

(Assuming that $\mathscr{G}_P$ is *not* a generator)

Goal: Find a circuit C' of small size such that

$$\begin{aligned} C' \circ \mathscr{G}_P &= 0 \\ (\partial_n C') \circ \mathscr{G}_P &\neq 0. \end{aligned}$$

If $\tilde{C}(x_n) = C(g_0, \ldots, g_{n-1}, x_n) \neq 0$

$$\begin{aligned} (\partial_n^r C)(g_0, \ldots, g_{n-1}, g_n) = 0 \\ (\partial_n^{r+1} C)(g_0, \ldots, g_{n-1}, g_n) \neq 0 \end{aligned}$$

Setting-up the initial conditions

(Assuming that $\mathscr{G}_P$ is *not* a generator)

Goal: Find a circuit C' of small size such that

$$C' \circ \mathscr{G}_P = 0$$
$$(\partial_n C') \circ \mathscr{G}_P \neq 0.$$

If $\tilde{C}(x_n) = C(g_0, \ldots, g_{n-1}, x_n) \neq 0$

$$(\partial_n^r C)(g_0, \ldots, g_{n-1}, g_n) = 0$$
$$(\partial_n^{r+1} C)(g_0, \ldots, g_{n-1}, g_n) \neq 0$$

$C' = (\partial_n^r C)$ is what we want.

Setting-up the initial conditions

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Goal: Find a circuit C' of small size such that

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If $\tilde{C}(x_n) = C(g_0, \ldots, g_{n-1}, x_n) \neq 0$

$$(\partial_n^r C)(g_0, \ldots, g_{n-1}, g_n) = 0$$
$$(\partial_n^{r+1} C)(g_0, \ldots, g_{n-1}, g_n) \neq 0$$

$C' = (\partial_n^r C)$ is what we want.

And, $\mathrm{size}(C') \leq \mathrm{size}(C) \cdot \deg(C)$

Some basic properties

$$\Delta_i(P) = \sum_{\mathbf{e}:|\mathbf{e}|=i} \frac{\mathbf{y}^{\mathbf{e}}}{\mathbf{e}!} \cdot (\partial_{\mathbf{e}} P)(\mathbf{z}).$$

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Additivity:

$$\Delta_i(P+Q) = \Delta_i(P) + \Delta_i(Q)$$

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$$\Delta_i(P) = \sum_{\mathbf{e}:|\mathbf{e}|=i} \frac{\mathbf{y}^{\mathbf{e}}}{\mathbf{e}!} \cdot (\partial_{\mathbf{e}} P)(\mathbf{z}).$$

Additivity:

$$\Delta_i(P+Q) = \Delta_i(P) + \Delta_i(Q)$$

‘Homogeneity’:

$$P(\mathbf{z}) = Q(\mathbf{z}) \bmod \langle \mathbf{z} \rangle^t$$
$$\implies \Delta_i(P) = \Delta_i(Q) \bmod \langle \mathbf{z} \rangle^{t-i}$$

Some basic properties

$$\Delta_i(P) = \sum_{\mathbf{e}:|\mathbf{e}|=i} \frac{\mathbf{y}^{\mathbf{e}}}{\mathbf{e}!} \cdot (\partial_{\mathbf{e}} P)(\mathbf{z}).$$

Additivity:

$$\Delta_i(P+Q) = \Delta_i(P) + \Delta_i(Q)$$

'Homogeneity':

$$P(\mathbf{z}) = Q(\mathbf{z}) \bmod \langle \mathbf{z} \rangle^t$$

$$\implies \Delta_i(P) = \Delta_i(Q) \bmod \langle \mathbf{z} \rangle^{t-i}$$

$$P = P_0 + \cdots + P_d$$

$$\Delta_i(P) = \Delta_i(P_{\leq t+i-1}) \bmod \langle \mathbf{z} \rangle^t$$

The Reconstruction Step

$$
\begin{aligned}
C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{z}) &= 0 \\
(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{z}) &\neq 0
\end{aligned}
$$

The Reconstruction Step

$$
\begin{aligned}
C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) &= 0 \\
(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) &\neq 0
\end{aligned}
$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

Else, replace $\langle z_1, \ldots, z_\ell \rangle$ with $\langle z_1 - \alpha_1, \ldots, z_k - \alpha_k \rangle$ in what follows

The Reconstruction Step

$$
\begin{aligned}
C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) &= 0 \\
(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) &\neq 0
\end{aligned}
$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = \underbrace{P_0 + \cdots + P_n}_{\text{Bruteforce in } n^{O(k)} \text{ size}} + P_{n+1} + \cdots + P_d$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = \underbrace{P_0 + \cdots + P_n}_{\text{Bruteforce in } n^{O(k)} \text{ size}} + \underbrace{P_{n+1} + \cdots + P_d}_{\text{Compute, via Newton iterations, one by one}}$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$C'\left(g_0, \ldots, g_{n-1}, g_n\right) = 0$$

The Reconstruction Step

$$C' \circ \mathcal{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathcal{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$C'(\Delta_0(P), \ldots, \Delta_{n-1}(P), \Delta_n(P)) = 0$$

The Reconstruction Step

$$\begin{aligned} C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) &= 0 \\ (\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) &\neq 0 \end{aligned}$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$C'(\Delta_0(P), \ldots, \Delta_{n-1}(P), \Delta_n(P)) = 0 \bmod \langle \mathbf{z} \rangle^2$$

The Reconstruction Step

$$\begin{aligned} C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) &= 0 \\ (\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) &\neq 0 \end{aligned}$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$(\Delta_i(P) = \Delta_i(P_{\leq t+i-1}) \bmod \langle \mathbf{z} \rangle^t)$$

$$C'(\Delta_0(P), \ldots, \Delta_{n-1}(P), \Delta_n(P)) = 0 \bmod \langle \mathbf{z} \rangle^2$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$(\Delta_i(P) = \Delta_i(P_{\leq t+i-1}) \bmod \langle \mathbf{z} \rangle^t)$$

$$C'\big(\Delta_0(P_{\leq n}), \ldots, \Delta_{n-1}(P_{\leq n}), \Delta_n(P_{\leq n+1})\big) = 0 \bmod \langle \mathbf{z} \rangle^2$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$(\Delta_i(P) = \Delta_i(P_{\leq t+i-1}) \bmod \langle \mathbf{z} \rangle^t)$$

$$C'\Big(\Delta_0(P_{\leq n}), \ldots, \Delta_{n-1}(P_{\leq n}), \Delta_n(P_{\leq n}) + \Delta_n(P_{n+1})\Big) = 0 \bmod \langle \mathbf{z} \rangle^2$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
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The Reconstruction Step

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$$C'(R_0, \ldots, R_{n-1}, R_n + A) = 0 \bmod \langle \mathbf{z} \rangle^2$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

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$$C'\left(\Delta_0(P_{\leq n}), \ldots, \Delta_{n-1}(P_{\leq n}), \Delta_n(P_{\leq n}) + \Delta_n(P_{n+1})\right) = 0 \bmod \langle \mathbf{z} \rangle^2$$

$$C'(R_0, \ldots, R_{n-1}, R_n + A) = 0 \bmod \langle \mathbf{z} \rangle^2$$

$$= C'(R_0, \ldots, R_{n-1}, R_n) + A \cdot ((\partial_n C')(R_0, \ldots, R_n)) = 0 \bmod \langle \mathbf{z} \rangle^2$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$(\Delta_i(P) = \Delta_i(P_{\leq t+i-1}) \bmod \langle \mathbf{z} \rangle^t)$$

$$C'\big(\Delta_0(P_{\leq n}), \ldots, \Delta_{n-1}(P_{\leq n}), \Delta_n(P_{\leq n}) + \Delta_n(P_{n+1})\big) = 0 \bmod \langle \mathbf{z} \rangle^2$$

$$C'(R_0, \ldots, R_{n-1}, R_n + A) = 0 \bmod \langle \mathbf{z} \rangle^2$$

$$= C'(R_0, \ldots, R_{n-1}, R_n) + A \cdot \big((\partial_n C')(R_0, \ldots, R_n)(\mathbf{y}, \mathbf{0})\big) = 0 \bmod \langle \mathbf{z} \rangle^2$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

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$$C'\Big(\Delta_0(P_{\leq n}), \ldots, \Delta_{n-1}(P_{\leq n}), \Delta_n(P_{\leq n}) + \Delta_n(P_{n+1})\Big) = 0 \bmod \langle \mathbf{z} \rangle^2$$

$$C'(R_0, \ldots, R_{n-1}, R_n + A) = 0 \bmod \langle \mathbf{z} \rangle^2$$

$$= C'(R_0, \ldots, R_{n-1}, R_n) + A \cdot \Big((\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0})\Big) = 0 \bmod \langle \mathbf{z} \rangle^2$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$(\Delta_i(P) = \Delta_i(P_{\leq t+i-1}) \bmod \langle \mathbf{z} \rangle^t)$$

$$C'\Big(\Delta_0(P_{\leq n}), \ldots, \Delta_{n-1}(P_{\leq n}), \Delta_n(P_{\leq n}) + \Delta_n(P_{n+1})\Big) = 0 \bmod \langle \mathbf{z} \rangle^2$$

$$C'(R_0, \ldots, R_{n-1}, R_n + A) = 0 \bmod \langle \mathbf{z} \rangle^2$$

$$= C'(R_0, \ldots, R_{n-1}, R_n) + A \cdot \Big((\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0})\Big) = 0 \bmod \langle \mathbf{z} \rangle^2$$

$$\therefore A = \left(\frac{C'(R_0, \ldots, R_n)}{(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0})} \right) \bmod (\mathbf{z})^2$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P\left(\mathbf{y}, \mathbf{0}\right) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P\left(\mathbf{y}, \mathbf{0}\right) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$\Delta_n(P_{n+1}) = \left(\frac{C'\left(\Delta_0(P_{\leq n}), \ldots, \Delta_n(P_{\leq n})\right)}{(\partial_n C') \circ \mathscr{G}_P\left(\mathbf{y}, \mathbf{0}\right)} \right) \bmod \langle \mathbf{z} \rangle^2$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$\Delta_n(P_{n+1})(\mathbf{a}, \mathbf{z}) = \left(\frac{C'\big(\Delta_0(P_{\leq n}), \ldots, \Delta_n(P_{\leq n})\big)(\mathbf{a}, \mathbf{z})}{(\partial_n C') \circ \mathscr{G}_P(\mathbf{a}, \mathbf{0})} \right) \bmod \langle \mathbf{z} \rangle^2$$

The Reconstruction Step

$$C' \circ \mathscr{G}_P\,(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P\,(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$\Delta_n(P_{n+1})(\mathbf{a}, \mathbf{z}) = \left(\frac{C'\big(\Delta_0(P_{\leq n}), \ldots, \Delta_n(P_{\leq n})\big)(\mathbf{a}, \mathbf{z})}{(\partial_n C') \circ \mathscr{G}_P\,(\mathbf{a}, \mathbf{0})} \right) \bmod \langle \mathbf{z} \rangle^2$$

By trying many $\mathbf{a}$'s, we can obtain all of $\partial^{=n}(P_{n+1})$

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$\Delta_n(P_{n+1})(\mathbf{a}, \mathbf{z}) = \left(\frac{C'\big(\Delta_0(P_{\leq n}), \ldots, \Delta_n(P_{\leq n})\big)(\mathbf{a}, \mathbf{z})}{(\partial_n C') \circ \mathscr{G}_P(\mathbf{a}, \mathbf{0})} \right) \bmod \langle \mathbf{z} \rangle^2$$

By trying many $\mathbf{a}$'s, we can obtain all of $\partial^{=n}(P_{n+1})$ and hence P_{n+1} itself

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$\Delta_n(P_{n+1})(\mathbf{a}, \mathbf{z}) = \left(\frac{C'\big(\Delta_0(P_{\leq n}), \ldots, \Delta_n(P_{\leq n})\big)(\mathbf{a}, \mathbf{z})}{(\partial_n C') \circ \mathscr{G}_P(\mathbf{a}, \mathbf{0})} \right) \bmod \langle \mathbf{z} \rangle^2$$

By trying many $\mathbf{a}$'s, we can obtain all of $\partial^{=n}(P_{n+1})$
and hence P_{n+1} itself

(Euler formula: $d \cdot f = \sum x_i \partial_i f$, if f homogeneous of degree d)

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$\Delta_n(P_{n+1})(\mathbf{a}, \mathbf{z}) = \left(\frac{C'\big(\Delta_0(P_{\leq n}), \ldots, \Delta_n(P_{\leq n})\big)(\mathbf{a}, \mathbf{z})}{(\partial_n C') \circ \mathscr{G}_P(\mathbf{a}, \mathbf{0})} \right) \bmod \langle \mathbf{z} \rangle^2$$

By trying many $\mathbf{a}$'s, we can obtain all of $\partial^{=n}(P_{n+1})$
and hence P_{n+1} itself

The Reconstruction Step

$$C' \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathscr{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$\Delta_n(P_{n+1})(\mathbf{a}, \mathbf{z}) = \left(\frac{C'\big(\Delta_0(P_{\leq n}), \ldots, \Delta_n(P_{\leq n})\big)(\mathbf{a}, \mathbf{z})}{(\partial_n C') \circ \mathscr{G}_P(\mathbf{a}, \mathbf{0})} \right) \bmod \langle \mathbf{z} \rangle^2$$

By trying many $\mathbf{a}$'s, we can obtain all of $\partial^{=n}(P_{n+1})$
and hence P_{n+1} itself
modulo higher order junk

The Reconstruction Step

$$C' \circ \mathcal{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathcal{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$\boxed{\Delta_n(P_{n+1})(\mathbf{a}, \mathbf{z}) = \left(\frac{C'\left(\Delta_0(P_{\leq n}), \ldots, \Delta_n(P_{\leq n})\right)(\mathbf{a}, \mathbf{z})}{(\partial_n C') \circ \mathcal{G}_P(\mathbf{a}, \mathbf{0})} \right) \bmod \langle \mathbf{z} \rangle^2}$$

By trying many $\mathbf{a}$'s, we can obtain all of $\partial^{=n}(P_{n+1})$
and hence P_{n+1} itself
~~modulo higher order junk~~

Border tricks!

Or careful homogenisation

The Reconstruction Step

$$C' \circ \mathcal{G}_P(\mathbf{y}, \mathbf{0}) = 0$$
$$(\partial_n C') \circ \mathcal{G}_P(\mathbf{y}, \mathbf{0}) \neq 0$$

$$P = P_0 + \cdots + P_n + P_{n+1} + \cdots + P_d$$

$$\boxed{\Delta_n(P_{n+j+1})(\mathbf{a}, \mathbf{z}) = \left(\frac{C'\big(\Delta_0(P_{\leq n+j}), \ldots, \Delta_n(P_{\leq n+j})\big)(\mathbf{a}, \mathbf{z})}{(\partial_n C') \circ \mathcal{G}_P(\mathbf{a}, \mathbf{0})} \right) \bmod \langle \mathbf{z} \rangle^{j+2}}$$

By trying many $\mathbf{a}$'s, we can obtain all of $\partial^{=n}(P_{n+j+1})$
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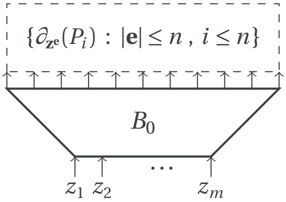
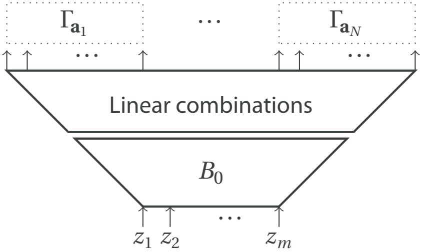
Reconstruction Step: Pictorially

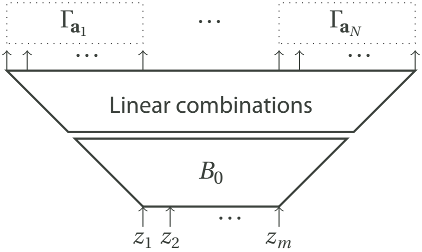


Reconstruction Step: Pictorially



Reconstruction Step: Pictorially

$$\Gamma_{\mathbf{a}} = \big(\Delta_0(P_{\leq n})(\mathbf{a},\mathbf{z})\,,\ \ldots\ ,\ \Delta_n(P_{\leq n})(\mathbf{a},\mathbf{z})\big)$$



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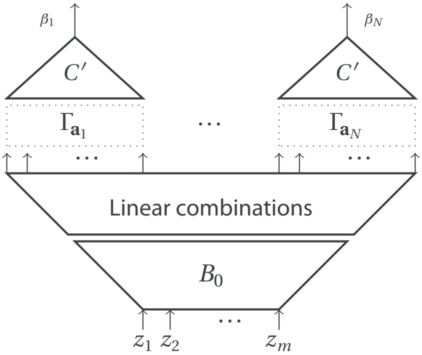

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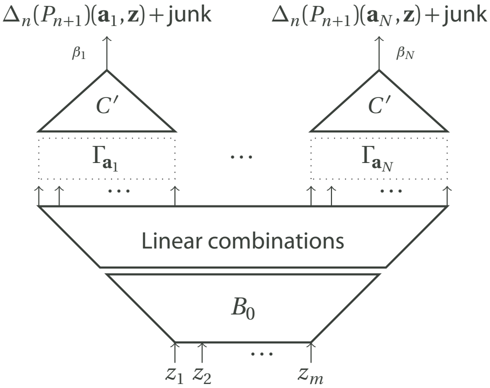
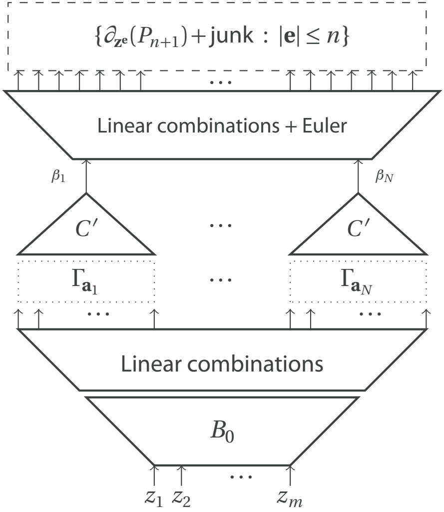


Reconstruction Step: Pictorially



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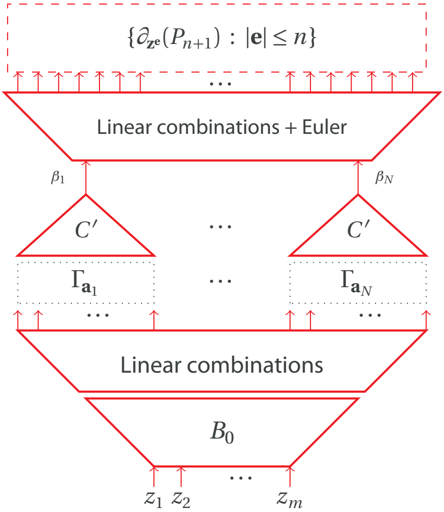

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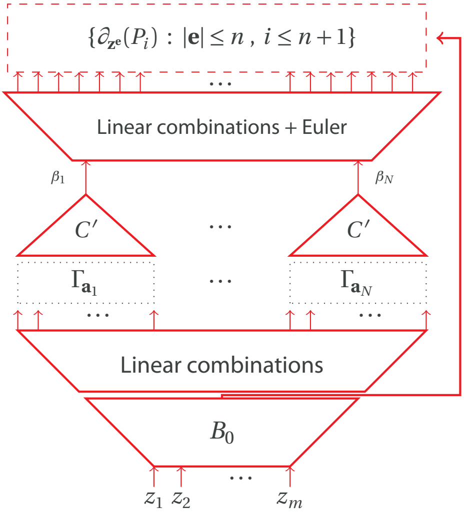
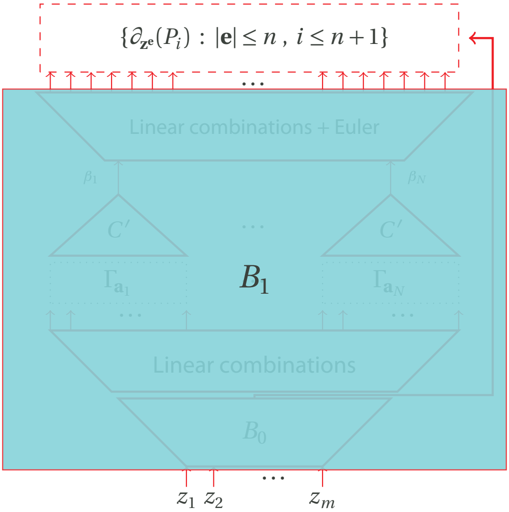
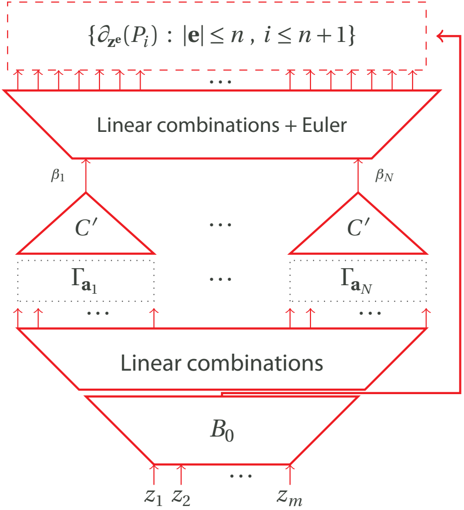


Reconstruction Step: Pictorially

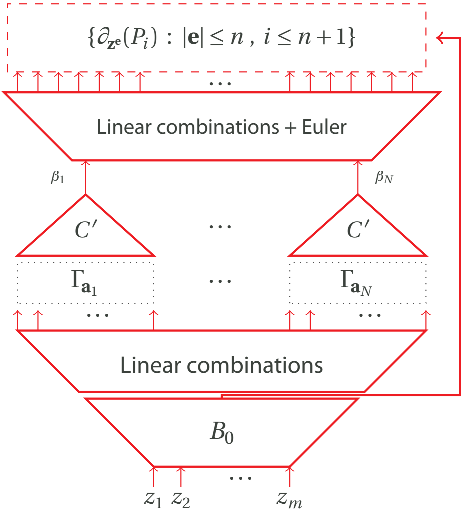


Reconstruction Step: Pictorially



$$s_0 = n^{O(k)}$$

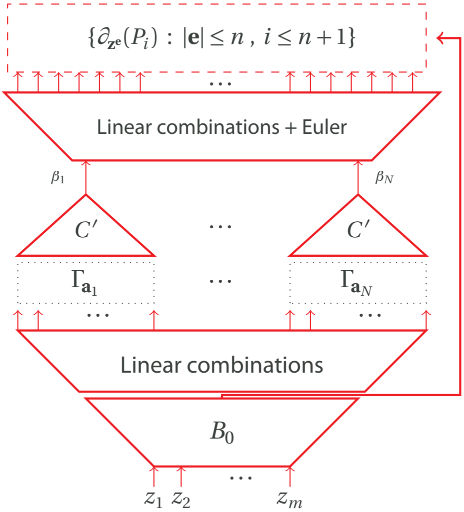
Reconstruction Step: Pictorially



$n^{O(k)}$

$s_0 = n^{O(k)}$

Reconstruction Step: Pictorially



$s' \cdot n^{O(k)}$

$n^{O(k)}$

$s_0 = n^{O(k)}$

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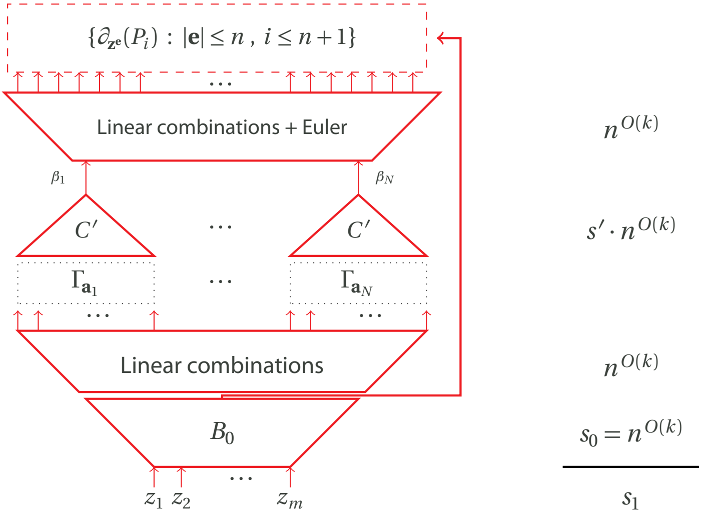

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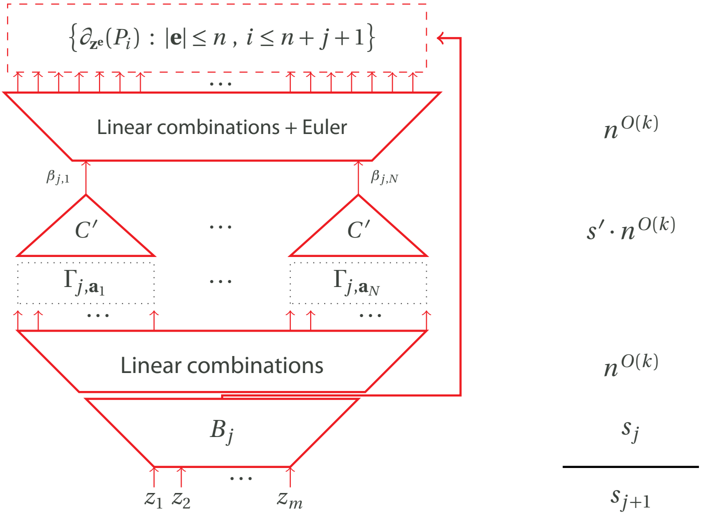

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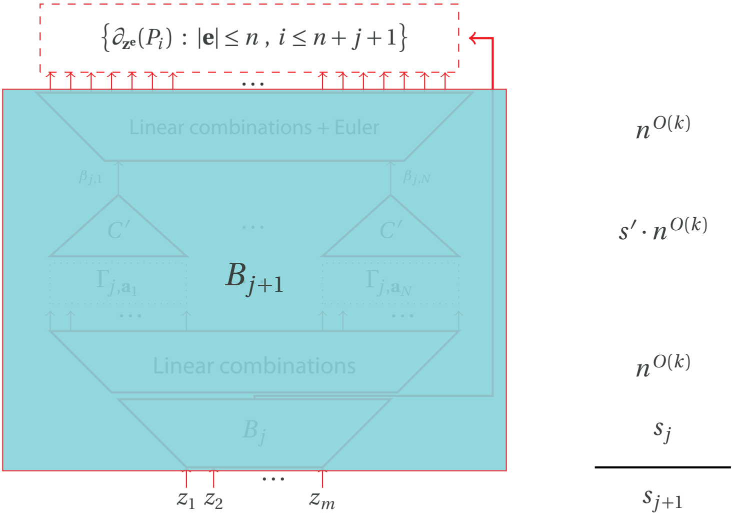

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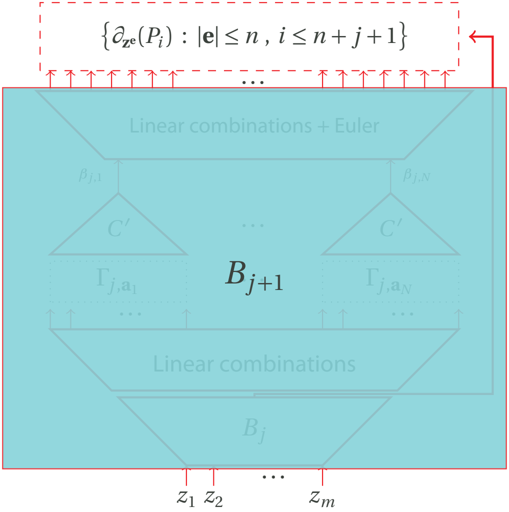


$n^{O(k)}$

$s' \cdot n^{O(k)}$

$n^{O(k)}$

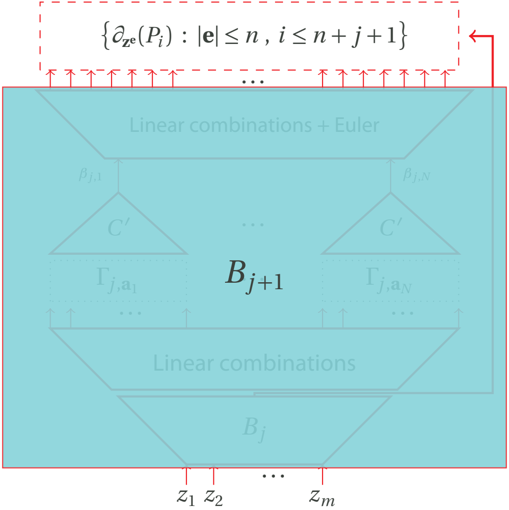
s_j

---

s_{j+1}

$$s_d \quad \leq \quad s' \cdot n^{O(k)} \cdot d$$

Reconstruction Step: Pictorially



$n^{O(k)}$

$s' \cdot n^{O(k)}$

$n^{O(k)}$

s_j

s_{j+1}

$$s_d \quad \leq \quad s \cdot D \cdot n^{O(k)} \cdot d$$

\end{proof}

Conclusion

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Summary:

- With suitable hardness, we can get poly-sized hitting sets.

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\end{document}