Krohn-Rhodes Theorem

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"Indeed there is no royal road through mathematics, but we do not need to break up the asphalt and destroy the signage to make travelling what roads there are a trial of one's skill." — Jordan Bell.

Contents

1 Partitions, Decompositions, and Automata

Let $A = (Q, \Sigma, f)$ be an automaton and $\mathcal{P}$ be a partition of Q. We say that $\mathcal{P}$ is **admissible** if for each $\sigma \in \Sigma$ and each $P \in \mathcal{P}$ we have $f_\sigma(P)$ is contained in some member of $\mathcal{P}$. If $\mathcal{P}$ is admissible, we naturally obtain an automaton $(\mathcal{P}, \Sigma, \bar{f})$ where $\bar{f}_\sigma(P)$ is defined as the (unique) member of $\mathcal{P}$ which contains $f_\sigma(P)$. We denote this automaton as $A/\mathcal{P}$.

A collection of subsets of a set Q is called a **decomposition** of Q if the union of all the elements of $\mathcal{D}$ is Q. Given an automaton $A = (Q, \Sigma, f)$ and a decomposition $\mathcal{D}$ of Q, we say that $\mathcal{D}$ is **admissible** if for each $D \in \mathcal{D}$ and each $\sigma \in \Sigma$ there is at least one $D' \in \mathcal{D}$ such that $f_\sigma(D)$ is contained in D'. Assuming $\mathcal{D}$ is admissible we may choose, for each $\sigma \in \Sigma$ and each $D \in \mathcal{D}$, a member D_σ of $\mathcal{D}$ such that $f_\sigma(D)$ is contained in D_σ. With this we obtain an automaton $(\mathcal{D}, \Sigma, \bar{f})$ where $\bar{f}_\sigma(D)$ is defined to be D_σ. When no confusion can occur, we denote the automaton thus obtained by $A/\mathcal{D}$, and will be referred to as a $\mathcal{D}$-**factor** of A.

Example 1.1. Let $A = (Q, \Sigma, f)$ be an automaton. Let k be an integer with $1 \leq k \leq |Q|$. Then the set of all the k-element subsets of Q forms an admissible decomposition of Q. ♢

1.1 Yoeli's Construction

Let $A = (Q, \Sigma, f)$ be an automaton and $\mathcal{D}$ be a decomposition of Q. Let $A/\mathcal{D} = (\mathcal{D}, \Sigma, \tilde{f})$ be a $\mathcal{D}$-factor of A. Define $Q_\mathcal{D} \subseteq Q \times \mathcal{D}$ as the set

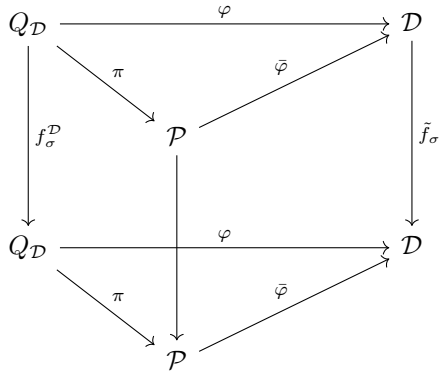
$$Q_\mathcal{D} = \{(q, D) : \ D \in \mathcal{D}, q \in D\} \tag{1.1}$$

For each $\sigma \in \Sigma$, define a map $f^\mathcal{D}_\sigma : Q_\mathcal{D} \to Q_\mathcal{D}$ which takes (q, D) to $(f_\sigma(q), \tilde{f}_\sigma(D))$. Then the triple $A_\mathcal{D} = (Q_\mathcal{D}, \Sigma, f^\mathcal{D})$ is an automaton and will be referred to as an **auxiliary semiautomaton** arising from A and $A/\mathcal{D}$. Let $\mathcal{P}$ be the partition of $Q_\mathcal{D}$ defined as $\mathcal{P} = \{D \times \{D\} : \ D \in \mathcal{D}\}$.

Lemma 1.2. *Then $A_\mathcal{D}/\mathcal{P}$ is isomorphic to $A/\mathcal{D}$.*

*For any questions, comments, suggestions, corrections, or typographical errors please write to me at this email address.

Proof. Let $\varphi : Q_{\mathcal{D}} \to \mathcal{D}$ be defined as $\varphi(q, D) = D$ for all (q, D) in $Q_{\mathcal{D}}$. Let $\pi : \mathcal{Q} \to \mathcal{P}$ be the natural map which sends an element of $Q_{\mathcal{D}}$ to the unique element of $\mathcal{P}$ that contains it. Then the following diagram commutes for all $\sigma \in \Sigma$.



where the vertical map from $\mathcal{P}$ to $\mathcal{P}$ is $\tilde{f}_\sigma^{\mathcal{D}}$ which sends $P \in \mathcal{P}$ to $f_\sigma^{\mathcal{D}}(P)$. Noticing that $\bar{\varphi}$ is bijective, we have the desired result. ■

2 Cascade Product

Let $A = (Q, \Sigma, f)$ and $A' = (Q', \Sigma', f')$ be two automata. A **connection mapping** from A to A' is a map $\omega : Q \to \mathrm{Maps}(\Sigma, \Sigma')$. The **cascade product** of A and A' with respect to a connection mapping ω is the automaton $A \circ_\omega A' = (Q \times Q', \Sigma, f^\omega)$ where f^ω is defined as

$$f_\sigma^\omega(q, q') = (f_\sigma(q), f'_{\omega_q(\sigma)}(q')) \tag{2.1}$$

A special case of this construction if the *direct product*, which is obtained as follows. Let $A = (Q, \Sigma, f)$ and $A' = (Q', \Sigma, f')$ be two automata over the same alphabet. Let $\omega : Q \to \mathrm{Maps}(\Sigma)$ be the trivial connection mapping, that is, ω_q is the identity map for each q. Then the transitions on $A \circ_\omega A'$ are given by

$$(q, q') \xrightarrow{\sigma} (f_\sigma(q), f'_\sigma(q')) \tag{2.2}$$

for each $\sigma \in \Sigma$, which justifies the name 'direct product.'

3 Coverings

Let $A = (Q, \Sigma, f)$ and $\tilde{A} = (\tilde{Q}, \tilde{\Sigma}, \tilde{f})$ be two automata. A **covering** from $\tilde{A}$ to A is a pair (φ, ξ), where $\varphi : \tilde{Q} \to Q$ is a surjective map and $\xi : \Sigma \to \tilde{\Sigma}$ such that the following diagram commutes

$$\begin{array}{ccc} \tilde{Q} & \xrightarrow{\varphi} & Q \\ \downarrow{\scriptstyle \tilde{f}_{\xi(\sigma)}} & & \downarrow{\scriptstyle f_\sigma} \\ \tilde{Q} & \xrightarrow{\varphi} & Q \end{array}$$

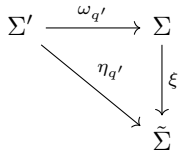
for each σ in Σ. We say that $\tilde{A}$ **covers** A if there is a covering from $\tilde{A}$ onto A. It is clear that if $\tilde{A}$ covers A and $\hat{A}$ covers $\tilde{A}$ then $\hat{A}$ covers A. In fact, if (φ, ξ) is a covering from $\tilde{A}$ onto A and (φ', ξ') is a covering from $\hat{A}$ onto $\tilde{A}$ then $(\varphi' \circ \varphi, \xi' \circ \xi)$ is a covering from $\hat{A}$ onto A.

Given an automaton $A = (Q, \Sigma, f)$, we say that a subset W of Q is **invariant** if $f_\sigma(W) \subseteq W$ for all $\sigma \in \Sigma$. By restricting the domain to f_σ to W, we obtain an automaton $(W, \Sigma, f|_W)$. A **partial covering** from $\tilde{A}$ onto A is a map (φ, ξ), where $\varphi : \tilde{W} \to Q$ is a surjective map, where $\tilde{W}$ is an invariant set in $\tilde{A}$, and ξ is a map $\xi : \Sigma \to \tilde{\Sigma}$. We say that $\tilde{A}$ **partially covers** A if there is a partial covering from $\tilde{A}$ onto A.

Lemma 3.1. *Let A, A' and $\tilde{A}$ be automata and suppose that $\tilde{A}$ covers A. Then whenever ω is a connection mapping from A' to A, there is a connection mapping η from A' to $\tilde{A}$ such that $A' \circ_\eta \tilde{A}$ covers $A' \circ_\omega A$.*

Proof. Let (φ, ξ) be a covering from $\tilde{A}$ onto A and $\omega : Q' \to \text{Maps}(\Sigma', \Sigma)$ be a connection mapping from A' to A. Define the map $\eta : Q' \to \text{Maps}(\Sigma', \tilde{\Sigma})$ as

$$\eta_{q'} = \xi \circ \omega_{q'} \tag{3.1}$$



Now define $\Phi : Q_\eta \to \mathbf{Q}_\omega$ and $\Xi : \Sigma' \to \Sigma'$ as

$$\Phi(q', \tilde{q}) = (q', \varphi(\tilde{q})) \qquad \Xi(\sigma') = \sigma' \tag{3.2}$$

for all $(q', \tilde{q}) \in Q^\eta$ and $\sigma' \in \Sigma'$[1], we see that (Φ, Ξ) is a covering from $A' \circ_\eta \tilde{A}$ onto $A' \circ_\omega A$. ■

Lemma 3.2. *Let $\mathcal{D}$ be an admissible partition of an automaton $A = (Q, \Sigma, f)$ and $A/\mathcal{D}$ be a $\mathcal{D}$-factor of A. Let $A_\mathcal{D}$ be the auxiliary automaton arising from A and $A/\mathcal{D}$. Then $A_\mathcal{D}$ covers A.*

Proof. Define $\varphi : Q_\mathcal{D} \to Q$ as $\varphi(q, D) = q$ for all (q, D) in $Q_\mathcal{D}$ and define $\xi : \Sigma \to \Sigma$ as the identity map. Then it is easy to check that (φ, ξ) is a covering map. ■

Lemma 3.3. *Let $\mathcal{P}$ be an admissible partition of an automaton $A = (Q, \Sigma, f)$ and $A/\mathcal{P}$ be the $\mathcal{P}$-factor of A. Let m be the maximum size of any element of $\mathcal{P}$. Then there is an automaton $A' = (Q', \Sigma', f')$ with $|Q'| \leq m$, and a connection mapping $\omega : \mathcal{P} \to \text{Maps}(\Sigma, \Sigma')$ such that $(A/\mathcal{P}) \circ_\omega A'$ partially covers A.*

Proof. Let $\mathcal{S}$ be a partition of Q with $|\mathcal{S}| = m$ such that the common refinement $\mathcal{P}$ and $\mathcal{S}$

$$\mathcal{P} \vee \mathcal{Q} = \{P \cap S : \ P \in \mathcal{P}, S \in \mathcal{S}\} \tag{3.3}$$

consists only of singletons, that is, $\mathcal{P} \vee \mathcal{S} = \{\{q\} : \ q \in Q\}$.[2] Fix S_0 in S arbitrarily and define the map $\pi_\mathcal{S} : Q \to \mathcal{S}$ which sends $q \in Q$ to the unique member of $\mathcal{S}$ which contains q. Also, we declare that $\pi_\mathcal{S}(\emptyset)$ is S_0.

Define $Q' = \mathcal{S}$, $\Sigma' = \mathcal{P} \times \Sigma$ and define, for each $(P, \sigma) \in \Sigma'$, the map $f'_{(P,\sigma)} : Q' \to Q'$ as

$$f'_{(P,\sigma)}(S) = \pi_\mathcal{S}(f_\sigma(P \cap S)) \tag{3.4}$$

Thus we have constructed an automaton $A' = (Q', \Sigma', f')$. For each $P \in \mathcal{P}$, define a map $\omega_P : \Sigma \to \Sigma'$ as

$$\omega_P(\sigma) = (P, \sigma) \tag{3.5}$$

Define $\omega : \mathcal{P} \to \text{Maps}(\Sigma, \Sigma')$ by declaring $\omega(P)$ to be ω_P. Thus ω is a connection mapping from $A/\mathcal{P}$ to A' and write $\tilde{A}$ to denote the corresponding cascade product. Let Q^* be a set of states in $\tilde{A}$ defined as

$$Q^* = \{(P, S) : \ P \cap S \neq \emptyset\} \tag{3.6}$$

Note that Q^* is an invariant set in $\tilde{A}$. Define a map $\varphi : Q^* \to Q$ by declaring $\varphi(P, S)$ to be the unique element of $P \cap S$, for all (P, S) in Q^*. Define $\xi : \Sigma \to \tilde{\Sigma} = \Sigma$ be the identity map. It is a routine check to verify that (φ, ξ) is a partial covering from $\tilde{A}$ onto A, finishing the proof. ■

Lemma 3.4. *Let $\mathcal{D}$ be an admissible decomposition of an automaton $A = (Q, \Sigma, f)$ and $A/\mathcal{D}$ be a $\mathcal{D}$-factor of A. Let m be the maximum size of any element of $\mathcal{D}$. Then there is an automaton $A' = (Q', \Sigma', f')$ with $|Q'| \leq m$, and a connection mapping $\omega : \mathcal{D} \to \text{Maps}(\Sigma, \Sigma')$ such that $(A/\mathcal{D}) \circ_\omega A'$ partially covers A.*

Proof. Let $A_\mathcal{D}$ be the auxiliary automaton arising from A and $A/\mathcal{D}$. By Yoeli's construction discussed in Section 1.1 we know that $A_\mathcal{D}/\mathcal{P}$ is isomorphic to $A/\mathcal{D}$, where $\mathcal{P} = \{D \times \{D\} : \ D \in \mathcal{D}\}$. By Lemma 3.3 we know that there is an automaton A' with

$$|Q'| \leq \max_{P \in \mathcal{P}} |P| = \max_{D \in \mathcal{D}} |D \times \{D\}| = \max_{D \in \mathcal{D}} |D| = m \tag{3.7}$$

such that a cascade product $(A_\mathcal{D}/\mathcal{P}) \circ_\omega A'$ partially covers $A_\mathcal{D}$. Since $A_\mathcal{D}$ covers A by Lemma 3.2, we deduce that $(A_\mathcal{D}/\mathcal{P}) \circ A'$ partially covers A. But since $A_\mathcal{D}/\mathcal{P}$ is isomorphic to $A/\mathcal{D}$, we conclude that a cascade product $(A/\mathcal{D}) \circ_\eta A'$ partially covers A and we are done. ■

[1] Thus Ξ is the identity map on Σ'.

[2] It is easy to argue that such a partition exists.

4 Permutation-Reset Automata

Let $A = (Q, \Sigma, f)$ be an automaton and $\sigma \in \Sigma$ be fixed. We say that σ is a **permutation input** if f_σ is a bijection. We say that σ is a **reset input** if the image of f_σ is a singleton. The automaton A is said to be a **permutation-reset automaton** if every σ in Σ is either a permutation input or a reset input. Similarly, A is said to be a **permutation automaton** if each input is a permutation input and a **reset automaton** if each input is a reset input.

Example 4.1. Let $A = (Q, \sigma, f)$ be an automaton and let $|Q| = n > 1$. Let $\mathcal{D}$ be the decomposition of Q consisting of all the $(n-1)$-elements subsets of Q. Then is an admissible decomposition. *We show that there is a $\mathcal{D}$-factor of A which is a permutation-reset automaton.* To see this, first note that if σ is a permutation input in A then it is also a permutation input in any $\mathcal{D}$-factor. For each such σ we set D_σ as $f_\sigma(D)$ for all $D \in \mathcal{D}$.

Now suppose σ is not a permutation input in A. Then f_σ is not a surjection, and hence we can find $D_\sigma \in \mathcal{D}$ such that $f_\sigma(D)$ is contained in D_σ for each D in $\mathcal{D}$. Now defining, for each σ in Σ, the map $\bar{f}_\sigma : \mathcal{D} \to \mathcal{D}$ which takes D to D_σ furnishes a permutation reset automaton $(\mathcal{D}, \Sigma, \bar{f})$. ♢

Theorem 4.2. *Let $A = (Q, \Sigma, f)$ be an automaton with $|Q| = n \geq 2$. Then A can be covered by a cascade product of at most $n-1$ permutation-reset automata.*

Proof. Let $\mathcal{D}$ be the decomposition of Q comprising of all the $(n-1)$-element subsets of Q. Then $\mathcal{D}$ is an admissible decomposition of Q. By Example 4.1 there is a $\mathcal{D}$-factor $A/\mathcal{D}$ which is a permutation-reset automaton. Now by Theorem 3.4, we know that there is an automaton A' with at most $n-1$ states and a connection mapping from $A/\mathcal{D}$ to A' such that $A/\mathcal{D} \circ_\omega A'$ covers A. Now we can finish inductively by making use of Lemma 3.1. ■

A Automata and Regular Languages

A **deterministic semiautomaton**, or DSA, is a triple $S = (Q, \Sigma, f)$, where Q is a finite set whose elements are called **states**, Σ is a finite set which we call the **alphabet**, and f is a map $f : \Sigma \to \text{Maps}(Q)$. We will usually write f_σ to denote the image of σ under f. Throughout this document we will write 'automaton' to mean a DSA.

Given two automata $A = (Q, \Sigma, f)$ and $A' = (Q', \Sigma, f')$ (they have the same alphabet) we say that A and A' are **isomorphic** if there is a bijective map $\varphi : Q \to Q'$ such that the following diagram commutes for each $\sigma \in \Sigma$.

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & Q' \\ \downarrow{\scriptstyle f_\sigma} & & \downarrow{\scriptstyle f'_\sigma} \\ Q & \xrightarrow{\varphi} & Q' \end{array}$$

Let $A = (Q, \sigma, f)$ be an automaton. Consider the monoid homomorphism $h : \Sigma^* \to \text{Maps}(Q)$ given by $\sigma \mapsto f_\sigma$. Let q_0 be a state in Q and w be a word in Σ^*. The **run** of the semiautomaton S on the word w starting at q_0 is defined as the state $q_0 h(w)$.

References