Quantum Theory without Classical Time: a Route to Quantum Gravity and Unification

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ABSTRACT

There must exist a reformulation of quantum field theory which does not employ classical time to describe evolution, even at low energies. To achieve this goal, we have proposed a pre-quantum, pre-spacetime theory, which is a matrix valued Lagrangian dynamics on an octonionic spacetime. This is a deterministic but non-unitary dynamics in which evolution is described by Connes time, a feature unique to non-commutative geometry. From here, quantum field theory and its indeterminism, as well as classical space-time geometry, are emergent under suitable approximations. In the underlying theory, the algebra of the octonions reveals evidence for the standard model of particle physics, and for its unification with a pre-cursor of gravitation, through extension to the Left-Right symmetric model and the symmetry group E_6. When elementary particles are described by spinors made from a Clifford algebra, the exceptional Jordan algebra yields a theoretical derivation of the low energy fine-structure constant, and of the observed mass ratios for charged fermions. We identify the Left-Right symmetry breaking with electroweak symmetry breaking, which also results in separation of emergent four-dimensional Minkowski spacetime from the internal symmetries which describe the standard model. This 'compactification without compactification' is achieved through the Ghirardi-Rimini-Weber mechanism of dynamical wave function collapse, which arises naturally in our theory, because the underlying fundamental Hamiltonian is necessarily non-self-adjoint. Only classical systems live in four dimensions; quantum systems always live in eight octonionic (equivalently ten Minkowski) dimensions. We explain how our theory overcomes the puzzle of quantum non-locality, while maintaining consistency with special relativity. We speculate on the possible connection of our work with twistor spaces and spinorial space-time, and with Modified Newtonian Dynamics (MOND). We point to the promising phenomenology of E_6, and mention possible experiments which could test the present proposal. In the end we outline further work that still remains to be done towards completion of this programme.

CONTENTS

I. OVERVIEW, AND SUMMARY OF THE MAIN RESULTS

*A Lagrangian for the standard model on a non-commutative spacetime: a paradigm shift. When we study the standard model using quantum field theory on a background spacetime, then the spacetime is a passive arena. QFT, dynamics and interactions are everything, understandably so, and very successfully so. However, when quantum gravity is incorporated into the standard model by making the background spacetime non-commutative (octonionic), the new space-time is no longer a *passive* arena. Even before QFT and interactions are switched on, the spacetime by itself determines the allowed properties of the elementary particles of the standard model. The spacetime dictates and determines the form of the allowed Lagrangian. In a coarse-grained average, the spacetime goes over to the classical spacetime of general relativity, and the dynamics goes over to conventional quantum field theory. This is a paradigm shift, namely that the geometry of non-commutative spacetime, and not the dynamics, determines the free parameters of the standard model of particle physics.*

The classical world consists of macroscopic bodies and classical fields, which give rise to the four dimensional space-time manifold, and its curved pseudo-Riemannian geometry. However, quantum theory, and classical space-time, do not go hand in hand, except as an approximation, even at the low energies prevailing in our present day universe. In principle, there need not be any classical objects even at low energy. Under such a situation, the collection of microscopic quantum systems making up the universe cannot give rise to a classical space-time manifold, nor

to its Einsteinian space-time geometry. How then do we describe the dynamics in such a universe? The present formulation of quantum field theory pre-assumes the existence of classical time, and hence of a universe dominated by macroscopic bodies. This is an approximate situation, not an exact one. Therefore there must exist a reformulation of quantum field theory which does not employ classical time to describe evolution. The development of such a reformulation is a route to quantum gravity and unification, as described in the present review. Conventional quantum field theory on a classical four-dimensional space-time background must emerge from the reformulation, as and when the universe gets dominated by macroscopic classical bodies.

This reformulation is arrived at in two steps. In the first step, given a classical Lagrangian for a set of dynamical variables, space-time is retained as Minkowski (or it could be a curved external space-time). Matter and field degrees of freedom, and their canonical momenta, are raised to the status of operators (equivalently matrices) but quantum commutation relations are not imposed. The Lagrangian is now a matrix polynomial, whose matrix trace defines a trace Lagrangian, whose space-time volume integral yields the action. We hence have at hand a matrix valued Lorentz invariant Lagrangian dynamics, known as Trace Dynamics, and developed by Stephen Adler and collaborators. The variation of the action with respect to the matrix valued configuration variables, and the use of a 'trace derivative', gives the matrix valued Lagrange equations of motion. A canonical trace Hamiltonian and matrix valued Hamilton's equations of motion are constructed; these equations take the place of Heisenberg's equations of motion. A global unitary invariance of the trace Hamiltonian implies the existence of a novel and important conserved charge, having dimensions of action, and known as the Adler-Millard charge. In general, the Hamiltonian of the theory will have an anti-self-adjoint part as well, and dynamical evolution will not be unitary.

From this trace dynamics, quantum field theory and its classical limit are derived as follows. It is assumed that the underlying dynamics operates at energy scales much higher than scales presently accessible in the laboratory; say the Planck scale. Equivalently, our present measurements are being made at time and length resolution scales much larger than Planck length and Planck time. We ask as to what the emergent dynamics is, if we coarse grain over time/length scales much larger than Planck scale. Using the methods of statistical thermodynamics it is shown that if the anti-self-adjoint part of the Hamiltonian is negligible, the Adler-Millard charge is equipartitioned over all degrees of freedom, quantum commutation relations emerge, and the statistically averaged dynamical variables obey Heisenberg equations of motion of quantum theory. A formal identification of thermodynamic averages of variables, with Wightman functions as they would be defined in quantum field theory [QFT], enables the recovery of QFT from trace dynamics.

If the anti-self-adjoint part of the Hamiltonian is not negligible, non-unitary evolution results in the breakdown of quantum superpositions, as in the Ghirardi-Rimini-Weber process of spontaneous localisation (dynamical collapse of the wave function). This in turn is responsible for the emergence of classical dynamics, as a limiting case of quantum theory. It can be shown that the anti-self-adjoint part of the Hamiltonian becomes significant only when sufficiently many degrees of freedom become entangled, so that the entangled system becomes macroscopic.

In the second step towards the reformulation, space-time degrees of freedom [manifold as well as metric] are raised to the status of matrices, by first rewriting the Einstein-Hilbert action in terms of the eigenvalues of the Dirac operator. This is the transition from geometry to algebra, the spectral action principle of Chamseddine and Connes. Each eigenvalue is raised to a matrix [thus making the transition to non-commutative geometry] which plays the role of a velocity in the Lagrangian, the corresponding configuration variable being the Yang-Mills field on the original manifold. Correspondingly, appropriate matrix variables are introduced for fermions, and for every Dirac eigenvalue we get one copy of the Lagrangian for an 'atom' of space-time-matter [STM atom]. The universe is made of enormously many STM atoms, and hence the total Lagrangian is the sum of that many copies of one atom's Lagrangian. Space-time is lost, however a new parameter, Connes time, which is a feature unique to non-commutative geometry, steps in to play the role of time and describe evolution. The action of the theory is the integral of the said Lagrangian over Connes time, and there are only three fundamental constants in the theory: Planck length, Planck time, and Planck's constant $\hbar$. In addition, the Lagrangian for each STM atom has two parameters: a length parameter, and a dimensionless coupling constant. We call this theory 'generalized trace dynamics'. The formalism of trace dynamics can be applied to obtain the Lagrange and Hamilton's equations of motion, as well as the Adler-Millard charge. This is the sought for pre-quantum, pre-spacetime theory, whose coarse-graining using the techniques of statistical thermodynamics as described above, gives the desired reformulation: quantum field theory without classical time. The reformulation can become necessary even at low energies, if the system under consideration is not dominated by classical subsystems (hence no classical spacetime): quantitatively speaking, the sub-systems have actions which do not satisfy the classicality requirement $S \gg \hbar$, and all subsystems are having sub-critical quantum entanglement. From here, one can recover four-dimensional classical space-time geometry and general relativity, and quantum field theory on a classical background.

Coming back to generalised trace dynamics, since this is a Lagrangian dynamics, it must provide a canonical definition of spin angular momentum, just as it defines canonical linear momentum and orbital angular momentum. Since spin cannot be related to motion in 4D space-time, we are

compelled to introduce internal directions and in fact double the dimensionality of space-time from four to eight, while also demanding that the new space-time be non-commutative. This naturally leads us to introduce the octonions, and consider generalised trace dynamics on an 8D octonionic space-time. Remarkably, the symmetries of the octonion algebra reveal the standard model of particle physics, and the possibility of a Kaluza-Klein type unification of internal symmetries with gravitation. We have a generalised matrix dynamics on an 8D octonionic space-time, evolving in Connes time. Quantum field theory is emergent; and very importantly, spontaneous localisation confines classical systems to four dimensional Minkowski space-time, their penetration depth into internal directions being less than a Planck length. However, quantum systems always stay in eight dimensions - this is what makes them quantum and non-classical - and the thickness of the extra dimensions is determined by the support of the wave function. The extra dimensions are never compactified in an ad hoc manner (compactification without compactification).

The introduction of the non-commutative octonionic space-time into particle physics opens up an entirely new paradigm in so far as properties of the elementary particles and the free parameters of the standard model are concerned. The internal symmetries can no longer be arbitrary, but are dictated by the octonion algebra to be precisely those of the standard model: $SU(3)_c \times SU(2)_L \times U(1)_{em}$. The spinorial states describing allowed fermions are constructed via a Clifford algebra (also known as geometric algebra) and quantization of electric charge is deduced. That there are three and exactly three fermion generations is argued for from the triality property of the group $SO(8)$. The automorphism group of the octonions very naturally suggests (through the split bioctonions, thus bringing in the right-handed sterile neutrino) an extension of the standard model to the Left-Right symmetric model and to the interpretation of the Right-handed part as 'would-be-gravity' $SU(3)_g \times SU(2)_R \times U(1)_g$, from the 'squaring' of which classical general relativity emerges. The associated symmetry groups are the exceptional Lie groups F_4 and E_6, with the former being the automorphism group of the exceptional Jordan algebra, whose characteristic equation is an ideal candidate for determining the free parameters of the standard model. By defining square-root of the mass of an elementary fermion as a $U(1)$ quantum number (also related to the Casimir of Poincare symmetry) we derive the observed mass ratios of the charged fermions, in the process showing that the neutrino is a Majorana fermion. In conjunction with the proposed Lagrangian in generalized trace dynamics, for describing the unification, we also derive the observed value of the low energy fine structure constant, from the eigenvalues of the exceptional Jordan algebra.

We identify the L-R symmetry breaking with the electro-weak symmetry breaking; prior to the breaking the universe has an E_6 symmetry. This is the automorphism group of the complexified

exceptional Jordan algebra, and it has promising phenomenology. There is no strong CP problem, a possible mechanism for the origin of matter-antimatter asymmetry, and the sterile neutrino possibly as a keV warm dark matter candidate. The theory also makes a few other predictions which can be tested in feasible experiments.

The would-be-gravity described by $SU(3)_g \times SU(2)_R \times U(1)_g$ shows a strong possibility of being the sought for relativistic MOND (Modified Newtonian Dynamics) theory. It comes with a naturally inbuilt critical acceleration scale whose origin lies in the cosmological expansion. In combination with warm dark neutrinos, the theory then offers a possible candidate for explaining CMB anisotropies, structure formation, and flat galaxy rotation curves.

We also remark on the possible connection that the octonionic spacetime of our theory bears with twistor spaces and with spinorial spacetime.

The generalized trace dynamics that we have proposed is a deterministic non-unitary theory. Unitary quantum evolution with its determinism, as well as quantum indeterminism and probabilities obeying the Born rule, are emergent consequences of the underlying dynamics. Through the extra dimensions (which are never compactified) the theory offers an elegant resolution of Einstein's EPR quantum non-locality puzzle, without getting into any conflict with 4D special relativity.

In the sections below we describe this research programme in some detail. We take care to emphasize that the concrete results to emerge so far include the symmetries of the standard model, the prediction of sterile neutrinos, the derivation of mass ratios of charged fermions, and the derivation of the low energy fine structure constant. The other ideas sketched above are in various stages of development and constitute work in progress. On the whole this appears to be a highly promising programme for unification of the standard model with gravitation, and for addressing the foundational problems of quantum theory. We have a precise starting point - a reformulation of quantum field theory without classical time - following this path has borne fruitful consequences, and it could be said that an exciting path lies ahead.

This research programme builds on the work of Ghirardi and collaborators on spontaneous localisation, of Adler and collaborators on trace dynamics, of Connes and collaborators on non-commutative geometry, and the extensive work of a large number of researchers on division algebras, Clifford algebras, and Jordan algebras, and their applications to the standard model.

II. WHY THE PRESENT FORMULATION OF QUANTUM FIELD THEORY IS APPROXIMATE, NOT EXACT, EVEN AT LOW ENERGIES?

Can we assume a four dimensional space-time manifold as a given, in which we then subsequently embed classical and quantum objects and fields, evolving according to known physical laws? The answer is no. In the absence of classical material bodies dominating the universe, it is not possible to operationally assign a physical meaning to distinct space-time points, and distinguish them from each other. This is a consequence of the Einstein hole argument, which concludes that giving a physical attribute to distinct space-time points requires the manifold to be overlaid by a dynamically determined metric tensor field. This field is produced by classical material bodies and classical fields, according to the laws of general relativity. Therefore, in order to meaningfully speak of the four dimensional space-time manifold, the universe must be dominated by classical macroscopic systems [1–3].

In principle, it is entirely possible that even at low energies, the universe is dominated, not by classical but by microscopic quantum systems which do not obey the classical dynamical laws of special and general relativity. The gravitational field produced by these quantum systems will possess ever-present quantum fluctuations, so that a definite metric overlying a chosen space-time point is no longer available. Hence the manifold is lost, and one does not anymore have a time parameter for describing evolution of quantum systems. For this reason, there must exist a reformulation of quantum field theory which does not depend on classical time. And such a reformulation must exist even at low energies. In particular the quantum field theory applied to the standard model of particle physics at currently accessible energies must have such a reformulation. When we do this, we are able to understand why the standard model has the observed properties and symmetries, and why the free parameters of the standard model (e.g. the low energy fine structure constant 1/137) take the values they do. This could be called the IR limit of the proposed quantum gravity and unification theory - one does not have to perform high energy experiments (or develop a high energy theory) at say the GUTs scale for understanding the standard model at low energies. Of course this same reformulation also informs us about phenomena at very high energies, say just after the big bang. Both the UV as well as the IR limit of the pre-quantum pre-spacetime theory get incorporated in the reformulation.

As an illustrative thought experiment, let us consider an electron in a double slit interference experiment, which has crossed the slits, but not yet reached the photographic screen. It is in a superposed state, as if it has passed through both the slits. We would like to know, non-

perturbatively, what is the space-time geometry produced by the electron? Additionally, let us imagine that every macroscopic object in the universe today is suddenly separated into its quantum elementary particle bits. Consequently we have lost classical space-time. Perturbative quantum gravity is no longer applicable. Nonetheless we ought to be able to describe the gravitational effect of the electron in its superposed state. This requires the sought for reformulation of quantum theory without classical time. And this quantum system is at low non-Planckian energies, and is even non-relativistic. This is the sought for reformulation we have developed, assuming only three fundamental constants a priori: Planck length L_P, Planck time t_P, and Planck's constant $\hbar$. Every other dimensionful constant, e.g. electric charge, and particle masses, are expressed in terms of these three. Note that Planck mass / energy is not amongst these three, and is a derivative constant in our theory.

A similar situation also arises in Penrose's criterion for breakdown of a quantum superposition of two different localised states of a massive object. As noted by Penrose, the superposition no longer permits a definite classical space-time geometry to be defined, and hence the time parameter describing evolution is also lost. The said reformulation of quantum theory becomes necessary in order to describe this physical system, which in fact is a quantum gravitational system. Since the energy scale involved is much lower than Planck energy scale, we now redefine as to under what conditions a system will be called quantum gravitational: possessing Planck scale energy will then be a sufficient but not necessary condition for gravity to be quantum.

Conventionally, a phenomenon is defined to be Planck scale if: the time scale T of interest is of the order Planck time t_P; and/or the length scale L of interest is of the order of Planck length L_P; and/or the energy scale E of interest is of the order Planck energy E_P. According to this definition of Planck scale, a Planck scale phenomenon is quantum gravitational in nature. Since the pre-theory is quantum gravitational, but not necessarily at the Planck energy scale, we must partially revise the above criterion, when going to the pre-theory, i.e. replace the criterion on energy E by a criterion on something else. This something is in fact the action of the system. In the pre-theory, a phenomenon is called Planck scale if: the time scale T of interest is of the order Planck time t_P; and/or length scale L of interest is of the order of Planck length L_P; and/or the action S of interest is of the order Planck constant $\hbar$. According to this new definition of Planck scale, a Planck scale phenomenon is quantum gravitational in nature. Why does this latter criterion make sense? If every (unentangled) degree of freedom has an associated action of order $\hbar$, together the many quantum degrees of freedom cannot give rise to a classical spacetime. Hence, even if the time scale T of interest and length scale L of interest are not Planck scale, this system

is quantum gravitational in nature. The associated energy scale $\hbar/T$ for each degree of freedom is much smaller than Planck scale energy E_P. Hence in the pre-theory the criterion for a system to be quantum gravitational is different from the one in conventional approaches to quantum gravity. Our criterion is more general (encompassing both the UV as well as IR limit) and includes the conventional criterion as a special case. And this makes all the difference to the formulation and interpretation of the theory. e.g. the low energy fine structure constant 1/137 is a Planck scale phenomenon [according to the new definition] because the square of the electric charge is order unity in the units $\hbar c = \hbar L_P/t_P$

A quantum system on a classical space-time background is definitely non-Planckian and not quantum gravitational. This is because the classical space-time is being produced by macroscopic bodies each of which has an action much larger than $\hbar$. The quantum system if treated in isolation is Planckian, but that is strictly speaking a very approximate description. The spacetime background cannot be ignored - when the background is removed from the description, an in principle requirement, the system is Planckian and quantum gravitational. Then one requires the pre-quantum, pre-spacetime theory, which is also quantum gravity.

The conventional procedure of quantisation works very successfully for non-gravitational interactions, because those are not concerned with space-time geometry. However, it might not be necessarily correct to apply this quantisation procedure to spacetime geometry, because the rules of quantum theory have been written by assuming a priori that classical time exists. How then can these quantisation rules be applied to classical time itself? As is well-known, doing so leads to the notorious problem of time in quantum gravity - time is lost, understandably. In our approach we do not quantise gravity. We remove classical space-time / gravity from quantum [field] theory. Space-time and gravity are emergent as approximations from the pre-theory, concurrent with the dominant emergence of classical macroscopic bodies. In the emergent universe, those systems which have not become macroscopic are described by the quantum field theory as we know it - namely quantum theory on a classical spacetime background. This is an approximation to the pre-theory: in this approximation, the contribution of the said quantum system to the background spacetime is [justifiably] neglected.

In the very early universe it is expected that classical space-time emerged from an earlier phase in which space-time was absent. Here of course the sought after reformulation of quantum field theory becomes essential. It is not reasonable to try and construct the primitive phase by quantising the classical phase which has emerged only subsequently. Furthermore, to define the time ordering of 'quantum gravitational phase before' to 'classical phase after' we need a new time parameter.

Fortunately, non-commutative geometry provides it, i.e. the Connes time.

We now describe the construction of the sought for reformulation, built in two steps: first trace dynamics, then generalised trace dynamics.

III. TRACE DYNAMICS: A PRE-QUANTUM THEORY ON MINKOWSKI SPACE-TIME

What is Trace Dynamics? : Trace dynamics is quantisation, without imposing the Heisenberg algebra: In the conventional development of canonical quantisation, the two essential steps are: (i) Quantisation Step 1 is to raise classical degrees of freedom, the real numbers q and p, to the status of operators / matrices. This is a very reasonable thing to do. (ii) Quantisation Step 2: Impose the Heisenberg algebra $[q,p] = i\hbar$. Its justification is that the theory it gives rise to is extremely successful and consistent with every experiment done to date. In classical dynamics, the initial values of q and p are independently prescribed. There is no relation between the initial q and p. Once prescribed initially, their evolution is determined by the dynamics. Whereas, in quantum mechanics, a theory supposedly more general than classical mechanics, the initial values of the operators q and p must also obey the constraint $[q,p] = i\hbar$. This could be considered highly restrictive?

It could be considered more reasonable if there were to be a dynamics based only on Quantisation Step 1. And then Step 2 emerges from this underlying dynamics in some approximation. This is precisely what Trace Dynamics is. Only step 1 is applied to classical dynamics. q and p are raised to matrices, and the Lagrangian is now the trace of a matrix polynomial made from q and its velocity. The matrix valued equations of motion then follow from variation of the trace Lagrangian. They describe dynamics and are a replacement for the Heisenberg equations of motion. This is the theory of trace dynamics developed by Adler [4–6] - a pre-quantum theory, which we have subsequently generalised to a pre-quantum, pre-spacetime theory [7]. This matrix valued dynamics, i.e. trace dynamics, is indeed more general than quantum field theory, and assumed to hold at the Planck scale. The Heisenberg algebra of quantum theory is shown to emerge at lower energies, after coarse-graining of the trace dynamics over length scales much larger than Planck length scale. Thus, quantum field theory can be thought of as being midway between trace dynamics and classical dynamics.

Suppose we consider classical dynamics [either Newtonian mechanics or special relativity] as our starting point, and instead of describing a material point particle by a real number, we choose to describe it by a matrix (equivalently, operator). This is indeed the essence of trace dynamics: for

a particle q, now described by a matrix $\mathbf{q}$, the action is transformed as in the following example:

$$S = \int dt \; [\dot{q}^2 - q^2] \quad \longrightarrow \int dt \; Tr \; [\dot{\mathbf{q}}^2 - \mathbf{q}^2] \tag{1}$$

After replacing the configuration variable q by a matrix, the scalar Lagrangian is constructed by taking a matrix trace of the operator polynomial, and then the scalar action is constructed, as usual, by integrating the Lagrangian over time. A general trace Lagrangian $\mathbf{L}$ is a function of the various configuration variables $\mathbf{q}_i$ and their time derivatives $\dot{\mathbf{q}}_i$, and is made from the trace of an operator polynomial $\mathcal{L}$. This construction can be generalized to field theory by raising the field value at each space-time point to a matrix, constructing an operator polynomial, taking its trace to form a Lagrangian density, and integrating over four-volume to get the action.

The Lagrange equations of motion are obtained by varying the action with respect to the operator $\mathbf{q}_i$. In order to vary the trace Lagrangian with respect to an operator, the notion of a trace derivative is introduced next. The derivative of the trace Lagrangian $\mathbf{L}$ with respect to an operator $\mathcal{O}$ in the polynomial $\mathcal{L}$ is defined as

$$\delta \mathbf{L} = Tr \frac{\delta \mathbf{L}}{\delta \mathcal{O}} \delta \mathcal{O} \tag{2}$$

This quantity, known as the trace derivative is obtained by varying $\mathbf{L}$ with respect to $\mathcal{O}$ and then cyclically permuting $\mathcal{O}$ inside the trace, so that $\delta\mathcal{O}$ sits to the right of the polynomial $\mathcal{L}$.

We assume that the matrix elements are complex valued Grassmann numbers, which can be further sub-divided into even grade and odd grade Grassmann numbers. Any Grassmann matrix can obviously be split into a sum of two matrices: the bosonic part (made of even grade elements) and the fermionic part (made of odd grade elements). Bosonic (fermionic) matrices describe bosonic (fermionic) fields, as in conventional quantum field theory. Thus, in trace dynamics there are present bosonic degrees of freedom $\mathbf{q}_B$ and fermionic degrees of freedom $\mathbf{q}_F$.

The Lagrange equations

$$\frac{d}{dt}\Big(\frac{\delta \mathbf{L}}{\delta \dot{\mathbf{q}}_i}\Big) - \Big(\frac{\delta \mathbf{L}}{\delta \mathbf{q}_i}\Big) = 0 \tag{3}$$

are used to obtain the operator equations of motion, and they also define the canonical momenta. The configuration variables and the momenta do not commute amongst each other in general, and the commutation relations are arbitrary. This is precisely what makes trace dynamics different

from both classical dynamics as well as from quantum theory. Apart from the trace Hamiltonian,

$$\mathbf{H} = \sum_i \mathrm{Tr}[p_{Fi}\dot{q}_{Fi}] + \sum_i \mathrm{Tr}[p_{Bi}\dot{q}_{Bi}] - \mathrm{Tr}\,\mathcal{L} \tag{4}$$

there is another conserved charge of great significance, the Adler-Millard charge, denoted $\tilde{C}$. This charge is a consequence of a global unitary invariance of the trace Lagrangian and the trace Hamiltonian. It is given by

$$\tilde{C} = \sum_{r\in B}[q_r, p_r] - \sum_{r\in F}\{q_r, p_r\} \tag{5}$$

[We shall henceforth drop the bold notation from the canonical variables, it being understood that we deal with matrix/operator valued canonical variables]. The Adler-Millard charge is unique to matrix dynamics, and plays a key role in emergence of quantum theory from trace dynamics. If the trace Hamiltonian is self-adjoint, then the Adler-Millard charge can be shown to be anti-self-adjoint. Were the trace dynamics Hamiltonian to have an anti-self-adjoint component, this conserved charge picks up a self-adjoint component - this will be important for us when we incorporate gravity in trace dynamics.

The Hamilton's equations of motion are given as follows

$$\frac{\delta \mathbf{H}}{\delta q_r} = -\dot{p}_r, \qquad \frac{\delta \mathbf{H}}{\delta p_r} = \epsilon_r \dot{q}_r \tag{6}$$

where $\epsilon_r = 1(-1)$ when q_r is bosonic(fermionic).

IV. RECOVERING QUANTUM FIELD THEORY, AND ITS CLASSICAL LIMIT, FROM TRACE DYNAMICS

Trace dynamics is Lorentz invariant, and is assumed to take place at the Planck energy scale. TD does not specify a particular form for the fundamental Lagrangian of nature, though we will choose a particular form below when we incorporate gravity into TD. Since the physical systems that we observe and experiment with, all operate at energy scales much lower than Planck energy and are not probed over Planck times, we ask the following question: what is the averaged description of trace dynamics, if we coarse grain the trace dynamics over time intervals much larger than Planck times? We can imagine that there are extremely rapid variations in the canonical variables over Planck time scales, but there is a smoothed out dynamics at lower energies, where the rapid

variations have been smeared over. The methods of statistical mechanics are applied, treating the underlying dynamics as 'microscopic' degrees of freedom, to show that the emergent coarse-grained dynamics is relativistic quantum (field) theory.

Adler starts by constructing the matrix dynamics phase space, with (the real and imaginary parts of) each element $(q_r)_{lm}$ of q_r being a (pair of) independent degrees of freedom in phase space, along with the matrix component (again real and imaginary part) $(p_i)_{mn}$ of the corresponding momentum. We shall use the symbol x to denote q or p. A measure $d\mu$ is defined in the phase space, as

$$(x_r)_{mn} = (x_r)^0_{mn} + i(x_r)^1_{mn}; \quad d\mu = \prod_A d\mu^A; \quad d\mu^A = \prod_{r,m,n} d(x_r)^A_{mn} \tag{7}$$

where $A = 0, 1$ and the components $(x_r)^A_{mn}$ are real numbers. This measure is conserved during evolution, and it obeys Liouville's theorem. Moreover, the measure is also invariant under infinitesimal operator shifts $x_r \to x_r + \delta x_r$.

Next, a phase space density distribution function $\rho[\{x_r\}]$ is defined in the matrix element phase space, which determines the probability of finding the system point in some particular infinitesimal volume in phase space. A canonical ensemble is constructed with a sufficiently large number of identical systems, each of which start evolving from arbitrary initial conditions in the phase space. It is assumed that over time intervals much larger than Planck time, the accessible region of the phase space [i.e. the region consistent with a conserved trace Hamiltonian and a conserved Adler-Millard charge] is uniformly occupied, and hence that the long time average [the coarse-grained dynamics] can be determined from the ensemble average at any one given time. This equilibrium dynamics is determined by maximising the Boltzmann entropy

$$\frac{S_E}{k_B} = -\int d\mu \; \rho \; \ln \rho \tag{8}$$

subject to the constraints that the ensemble-averaged trace Hamiltonian $\langle \mathbf{H} \rangle_{AV}$ and the ensemble averaged Adler-Millard charge $\langle \tilde{\mathbf{C}} \rangle_{AV}$ are conserved. These two constraints are imposed by introducing Lagrange multipliers $\overline{\tau}$ and $\tilde{\lambda}$ respectively, where $\overline{\tau}$ is a constant with dimensions of inverse mass, and $\overline{\lambda}$ an anti-self-adjoint matrix with dimensions of inverse action.

Hence the phase space density distribution ρ depends, apart from dynamical variables, on $\tilde{C}, \tilde{\lambda}, \mathbf{H}, \overline{\tau}$ and can be written as $\rho(\tilde{C}, \tilde{\lambda}, \mathbf{H}, \overline{\tau})$. It can be shown that the dependence on $\tilde{C}$ and $\tilde{\lambda}$ is of the form $Tr(\tilde{\lambda}\tilde{C})$, so we write $\rho = \rho(Tr[\tilde{\lambda}\tilde{C}], \overline{\tau}, \mathbf{H})$. It can be proved, subject to the assumption

that the ensemble does not favour any one state in the ensemble over the other, that the canonical ensemble average of the Adler-Millard charge takes the form

$$\langle\tilde{C}\rangle_{AV} = i_{eff}\hbar; \qquad i_{eff} = i\ diag(1,-1,1,-1...,1,-1) \tag{9}$$

where the real constant $\hbar$ is eventually identified with Planck's constant, subsequent to the emergence of quantum dynamics.

The equilibrium distribution is found, as is standard, by maximising the function $-\mathcal{F}$ where

$$\mathcal{F} = \int d\mu\ \rho\log\rho + \theta\int d\mu\ \rho + \int d\mu\ \rho Tr\tilde{\lambda}\tilde{C} + \overline{\tau}\int d\mu\ \rho\mathbf{H} \tag{10}$$

and gives the result

$$\rho = Z^{-1}\exp\left(-\mathrm{Tr}\tilde{\lambda}\tilde{C} - \tilde{\tau}\mathbf{H}\right) \tag{11}$$

$$Z = \int d\mu\ \exp\left(-\mathrm{Tr}\tilde{\lambda}\tilde{C} - \tilde{\tau}\mathbf{H}\right) \tag{12}$$

The entropy of the system at equilibrium is given by

$$\frac{S_E}{k_B} = \log Z - \mathrm{Tr}\tilde{\lambda}\frac{\partial\log Z}{\partial\tilde{\lambda}} - \tilde{\tau}\frac{\partial\log Z}{\partial\tilde{\tau}} \tag{13}$$

What is the mean dynamics obeyed by the variables $\langle x\rangle_{AV}$, averaged over the canonical ensemble, at energy scales below Planck scale? To answer this question, one derives certain Ward identities, just as is done for functional integrals in quantum field theory, in analogy with the proof for the equipartition theorem in statistical mechanics. These identities are a consequence of the invariance of the phase space measure under constant shifts of the dynamical variables. Therefore, in conventional statistical mechanics, the equipartition theorem is a consequence of the vanishing of the integral of a total divergence:

$$0 = \int d\mu\ \frac{\partial[x_r\exp(-\ \beta H)]}{\partial x_s} \tag{14}$$

In the statistical mechanics of trace dynamics, for a general operator $\mathcal{O}$, its average over the canonical ensemble is unchanged when a dynamical variable is varied:

$$0 = \int d\mu\ \delta_{x_r}(\rho\mathcal{O}) \tag{15}$$

One chooses $\mathcal{O}$ to be the operator $Tr\{\tilde{C}, i_{eff}\}W$ where W is any bosonic polynomial function of the dynamical variables, and carries out the above variation, taking ρ to be the equilibrium phase space density distribution function. Thus we have

$$0 = \int d\mu \ \delta_{x_r} \ \left[\exp\left(-Tr\tilde{\lambda}\tilde{C} - \tilde{\tau}\mathbf{H}\right) \ Tr\{\tilde{C}, i_{eff}\}W\right] \tag{16}$$

An important assumption is made, namely that $\tilde{\tau}$ is the Planck time scale, and that we are actually interested in the averaged dynamics over much larger time scales (equivalently much lower energies). Every dynamical variable x_r is split into a 'fast' varying part [which varies over Planck times] and a 'slow' part which is constant over Planck times. Important conclusions then follow from the above Ward identity, by making different choices for W. When W is chosen to be a dynamical variable x_r, standard quantum commutation relations for bosonic and fermionic degrees of freedom are shown to be obeyed by the averaged variables $\langle x_r\rangle_{AV}$. The constant $\hbar$ introduced above is identified with Planck's constant. If W is identified with the operator polynomial H whose trace is the trace Hamiltonian $\mathbf{H}$, the quantum Heisenberg equations of motion for the averaged dynamical variables are obtained. The underlying matrices of trace dynamics, within ensemble averages, obey properties analogous to those of quantum fields. The contact with quantum field theory is then made as follows. There is a unique eigenvector ψ_0 whose corresponding eigenvalue is the lowest eigenvalue of H. This acts as the conventional vacuum state, and canonical ensemble averages are identified with Wightman functions in the emergent quantum field theory, for a given function S,

$$\psi_0^\dagger \ \langle S\{x_r\}\rangle_{AV} \ \psi_0 = \langle vac|S\{X\}|vac\rangle \tag{17}$$

where X is a quantum field operator. In this way, relativistic quantum (field) theory is shown to be an emergent phenomenon, being the low energy equilibrium approximation in the statistical thermodynamics of an underlying matrix dynamics. Once the Heisenberg equations of motion are known, one can also transform to the Schrodinger picture in the standard manner.

Trace dynamics also provides a theoretical basis for the origin of the phenomenological theory of continuous spontaneous localisation. As we have seen above, quantum dynamics is a mean dynamics arising from averaging over Planck time scales, and by neglecting the fast component in the variation of the dynamical variables. Under certain circumstances, the fast component can become significant, in which case its impact on the coarse-grained dynamics can be modelled

as stochastic fluctuations around equilibrium. Particularly crucial is that these fluctuations can make an ant-self-adjoint stochastic contribution to the quantum theory Hamiltonian. This can occur because the underlying trace Hamiltonian can have a small anti-self-adjoint part at the Planck scale, which could get amplified by entanglement between a very large number of particles. Precisely such a situation arises when gravity is included in trace dynamics, as we will see below.

Adler considers such a possibility for fermions, in the non-relativistic approximation to quantum field theory, where the anti-Hermitean fluctuating correction to the Hamiltonian is modelled by adding a stochastic function $\mathcal{K}(t)$

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi + i\mathcal{K}(t)\Psi \tag{18}$$

This modified equation does not preserve norm of the state vector during evolution. If we insist on norm-preservation, and transform to a new state vector whose norm is preserved, the resulting evolution equation is non-linear. It also makes the evolution non-unitary; if we also demand that the non-linear evolution should not lead to superluminal signalling, the form of the evolution becomes just the same as in spontaneous localisation models. Thus trace dynamics can in principle explain the quantum-to-classical transition, by taking into account the potential role of statistical fluctuations around equilibrium. The theory provides a common origin for quantum theory, as well as for spontaneous localisation, starting from an underlying matrix dynamics possessing a global unitary invariance.

The theory of trace dynamics does not specify the fundamental Lagrangian for physical interactions. Also, it does not include gravity, although it operates at the Planck scale. The theory also leaves a few important questions unanswered. What is the origin of the small anti-self-adjoint component of the Hamiltonian at the Planck scale? Why does spontaneous localisation take place only for fermions, but not for bosons? Why should the norm of the state vector be preserved despite the presence of the anti-Hermitean fluctuations? In the next section, we demonstrate how to include gravity in trace dynamics, using the principles of Connes' non-commutative geometry - this leads us to a candidate quantum theory of gravity, for which we specify a Lagrangian. We also make some progress towards answering the open questions left unanswered by trace dynamics, as mentioned in the previous lines.

V. GENERALIZED TRACE DYNAMICS: REMOVING CLASSICAL SPACE-TIME FROM TRACE DYNAMICS, AND INTRODUCING CONNES TIME

In order to remove classical space-time geometry from trace dynamics, we need to identify appropriate dynamical variables for gravity, which can then be raised to the status of matrices. The metric by itself will not do in this role, because the Einstein hole argument does not permit an operational meaning for the points in the underlying space-time manifold unless it is overlaid by a classical (non-operator valued) metric. Hence we have to remove the underlying spacetime manifold as well; individual spacetime points also have to acquire the status of a matrix, so to say, and the most direct option appears to be to appeal to Connes' non-commutative geometry, where points are indeed raised to the status of operators.

Connes' celebrated Reconstruction Theorem – the recovery of an underlying manifold from an algebra of functions on it – builds upon the work of Gel'fand and Naimark, who showed that there is a complete isometry between the geometry of a topological space (metric space, phase space, etc.) and the (commutative) C* algebra of continuous functions $f : M \to \mathbb{C}$. In other words, there is in a sense an equivalence, or rather *duality*, between the geometrical and algebraic descriptions [8]. This is of course useful in the case of quantum mechanics, where the spectra of observables is given by the operator algebra rather than a set of points on a classical spacetime. These operator algebras (Hilbert space operators, von Neumann algebras) are, importantly, non-commutative. Connes' non-commutative geometry (NCG) represents a greatly successful programme of geometrising the algebra of non-commutative operators, in effect asking the question: "In the spirit of Gel'fand, can we recover the unique and completely isometric geometrical space defined by the non-commutative C* algebra of bounded operators acting on a Hilbert space?"

Connes invents various algebraic-geometric tools to describe this new non-commutative "space" under consideration, including an abstract non-commutative (pseudodifferential) calculus, a method of Dixmier traces to solve integrals, and crucially, the notion of a *spectral triple*. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a Fredholm module that consists of a unital, involutive algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$, and a linear Hermitian operator $\mathcal{D}$ acting on $\mathcal{H}$. Connes' Reconstruction Theorem demonstrates that out of a commutative spectral triple (a spectral triple with a commutative algebra $\mathcal{A}$), one can recover the underlying manifold, with associated vector bundles and connections.

With a seminal contribution to the physics and philosophy of the notion of "distance", Connes (and Chamseddine) were able to demonstrate the Spectral Action principle – that the spectrum of

D – suitably defined – uniquely fixes the physical action. With the explicit definition of $ds = D^{-1}$ (where D is the Dirac operator) and the universally true commutation relation $[[D, f], g] = 0$ (for all $f, g \in \mathcal{A}$) Connes laid down the first two axioms of his geometry (thus far, we have only satisfied the requirements for a *commutative* geometry); other axioms include mathematical considerations such as smoothness and finiteness [9]. For the full departure into NCG, the second of these is modified with g^0 replacing g where g^0 is the canonical antilinear isometric involution $Jg^*J^{-1} = g^0$ from Tomita-Takesaki theory (note, f and g^0 commute).

The Tomita-Takesaki theorem [10, 11] is a powerful and relevant result that does not exist in the commutative analogue, indeed, it requires that the geometry be non-commutative. It asserts that there is a (as Connes puts it, "god-given") one-parameter group of inner automorphisms of the algebra $\mathcal{A}$ which gives us a universal parameter according to which non-commutative spaces evolve – which Connes identifies with evolution time. As we will see, this becomes important when we seek to work with matrix- or operator-time (such as in Trace Dynamics or Horwitz-Stuckelberg theory), where the evolution equations require a universal (and otherwise unmotivated) τ parameter. We shall call this Connes time τ. For an elaboration by Connes on this time parameter please see also this link.

The crucial physical insight offered by Connes is the new measure of distance – in essence, this is where the non-commutativity truly comes in. The identification of the distance measure ds as the inverse of the Dirac operator D with $D = \gamma^\mu \bigtriangledown_\mu$ represents a novel – and *physical* – measure of distance on a geometry (and well-motivated by physical intuition – this is nothing but the Feynman propagator) and creates tension between $ds = D^{-1}$ and the spatial coordinates $a \in A$. To quote Connes [12] –"it is precisely this lack of commutativity between the line element and the coordinates on a space that will provide the measure of distance". With Connes' redefinition of the differential (motivated by the quantum evolution equation), we now have $df = [D, f]$ giving the differential for any $f \in A$ in the NCG scheme.

In this scheme, the geodesic distance between two points (x, y) on a manifold is formally rewritten as [8]:

$$d(x, y) = \mathrm{Inf} L(\gamma) = \mathrm{Inf} \int_x^y ds = \mathrm{Inf} \int_x^y \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$$

where the $L(\gamma)$ are the lengths of the paths $\gamma : x \to y$. From this, it can then be shown (Connes's 'distance formula') that the dual algebraic version of this definition is in terms of the supremum

on $x \in M$ of the norms on the tangent spaces $T_x M$:

$$d(x, y) = \text{Sup}\{|f(y) - f(x)|; f \in A, ||\bigtriangledown f||_\infty \leq 1\}$$

The $\bigtriangledown$'s, as we have seen, are simply given by the commutators, $[D, f]$, and hence this definition helps us define the metric through D.

A note to be made here – due to the construction of the abstract geometry from the operator algebra on the basis of the spectral triple, $\{\mathcal{A}, \mathcal{H}, \mathcal{D}\}$, we have that the Riemannian manifold of space-time is no longer a fundamental entity, but rather, a derived concept, constructed out of the D^{-1} which smoothly cover ordinary space. This means that we no longer consider the configuration- or coordinate-space description as fundamental, indeed the spatial points $\{\mathbf{x}\}$ no longer have any meaning outside of the spectra of the corresponding operator in our algebra (such as $\hat{x}$). In other words, the $\{\mathbf{x}\}$ are merely a collection of eigenvalues, and are no longer to be used in the classical sense to define intervals $\mathbf{\Delta x} = \mathbf{x_2} - \mathbf{x_1}$. This means, among other things, that nowhere in the NCG scheme do we refer to the commutativity of the coordinates themselves, they could indeed commute or not commute depending on the specific theory at hand. Every spatial interval ds is given by the commutator with the Dirac operator, $[D, x]$, while the relativistic line element is simply in terms of these commutators and the metric (at this stage no more than a matrix of numbers): $ds^2 = [D, x^\mu] * g_{\mu\nu}[D, x^\nu]$.

In effect, the only measure of distance here is the commutator with the Dirac operator. With this measure of distance, we have the important result in NCG [9]: If $A = C^\infty M$ where M is a smooth, compact manifold and all the NCG axioms are satisfied, then there exists a unique Riemannian metric g on M defined as above, and

$$⨍ ds^2 = \frac{-1}{48\pi^2} \int_{M4} R\sqrt{g} d^4x \tag{19}$$

Connes motivates this result on symmetry grounds, while Kalau & Walze, and separately, Kastler, demonstrates this explicitly, using the Wodzicki residue. The crucial point here is that the spectral action can be formulated as an expansion. As mentioned, the spectral action is a functional on the space of spectral triples. More explicitly, it is a regularised heat kernel expansion of D. As the Einstein-Hilbert action is a functional on the space of Riemannian manifolds, the spectral action is a functional on a general non-commutative space. For ordinary Riemannian spaces, the spectral

action reduces to the Einstein-Hilbert action, plus integrals over higher curvature invariants.

Two different mathematical tools were unified by Connes, in his attempt to calculate integrals of members of the operator algebra, $T \in A$, the Dixmier trace and the Wodzicki residue. The integral $\fint T$ is defined as the integral of a first-order infinitesimal in T, and is identified with the coefficient of logarithmic divergence in the trace of T. This is due to the standard analysis by Dixmier [12], which allows us to compute singular traces on a space of linear operators. This can be applied to the *dimension spectrum* of a non-commutative geometry, defined as the subset of the complex plane $\mathcal{C}$ where the spectral functions have singularities. Assuming the spectral functions have at most simple poles, the residues at the poles are given by the Dixmier traces. This is simply an extension of the Wodzicki residue of pseudodifferential operators on manifolds. In other words,

$$\fint T = \mathrm{Res}_{s=0} Tr(T|D|^{-s})$$

giving us the general method of calculating integrals in the NCG framework [13].

Kalau and Walze [14] successfully use the method of calculating Wodzicki residues to derive the Einsein-Hilbert action. From the observation that the metric structure is fully encoded in the Dirac operator of a K-cycle, they set out to define the curvature tensors and the EH action from the logarithmic divergent part of $\mathrm{tr}(D^{-n+2})$. They find that the Wodzicki residue of D^{-n+2} picks out the second heat kernel coefficient and results in the EH action. Kastler [15] derives independently the same result. Both parties start with the Lichnerowicz formula, which expresses the Dirac operator in terms of the Clifford connection, the metric, and the scalar curvature [16]:

$$D^2 = -g_{\mu\nu}(\tilde{\bigtriangledown}_\mu \tilde{\bigtriangledown}_\nu - \Gamma^\alpha_{\mu\nu} \tilde{\bigtriangledown}_\alpha) + \frac{1}{4} R \tag{20}$$

After some calculation (see [15] and [14]), this yields the desired result $\fint ds^2 = S_{GR} = (1/12) \int d^n x \sqrt{g} R$. The details of the derivation are somewhat tedious, but it suffices to note that the first group of terms end up being proportional to the scalar curvature as well, allowing us to end up with an overall multiple of the curvature.

In the context of the standard model of particle physics coupling to gravity, the spectral action of the gravity sector can be written as a simple function of the square of the Dirac operator, using a cut-off function $\chi(u)$ which vanishes for large u [17] (and references therein)

$$S_G[D] = \kappa Tr[\chi(L_P^2 D^2)] \tag{21}$$

The constant κ is chosen so as to get the correct dimensions of action, and the right numerical coefficient.

At curvature scales smaller than Planck curvature, this action can be related to the Einstein-Hilbert action using the heat kernel expansion:

$$S_G[D] = L_P^{-4} f_0\ \kappa \int_M d^4x\ \sqrt{g} + L_p^{-2} f_2 \kappa \int_M d^4x \sqrt{g} R + ... \tag{22}$$

Here, f_0 and f_2 are known functions of χ and the further terms which are of higher order in L_p^2 are ignored. Also, we will not consider the cosmological constant term for the purpose of the present discussion, and assume the cut-off function as implied, without explicitly writing it every time. We will subsequently below argue as to why it might be entirely drop the cosmological constant term - cancellation of bosonic and fermionic zero point energies makes it exactly zero, and there is an alternative explanation for dark energy. Furthermore, it might also be entirely possible to drop the higher curvature terms exactly, because this theory will be formulated on an octonionic spacetime, and in constructing a projective plane from the exceptional Jordan algebra, there is no projective plane beyond $\mathbb{OP}^2$ ($n > 2$ does not exist for $\mathbb{OP}^n$ unlike for other division algebras). (Put differently, there are only finitely many exceptional Lie groups, with E_8 being the last one).

Let us compare and contrast the above definition of spectral action with how a trace action is defined in Adler's theory of trace dynamics [6]. This spectral action, which includes the Einstein-Hilbert action as a part of its heat kernel expansion, is the trace of an operator, namely D^2. Comparing however with trace dynamics, we see the difference in the two cases: in trace dynamics it is the Lagrangian [not the action] which is made of trace of a polynomial. Thus, the way things stand, we cannot use the spectral action directly in trace dynamics to bring in gravity into matrix dynamics. We need to think of the spectral action as a Lagrangian, and we then need to integrate that Lagrangian over time, to arrive at something analogous to the action in trace dynamics. We can convert the spectral action into a quantity with dimensions of a Lagrangian, simply by multiplying it by c/L_p. But which time parameter to integrate the Lagrangian over? The space-time coordinates have already been assumed to be non-commuting operators, especially in the definition of the atom of space-time-matter, the case that we are interested in. So it seems as if we have a Lagrangian, but we do not have a time parameter over which to integrate the Lagrangian, so as to make an action.

Fortunately, non-commutative geometry itself comes with a ready-made answer. The required time parameter is the Connes time τ. As we saw, in NCG, according to the Tomita-Takesaki

theorem, there is a one-parameter group of inner automorphisms of the algebra $\mathcal{A}$ of the non-commuting coordinates - this serves as a 'god-given' (as Connes puts it) time parameter with respect to which non-commutative spaces evolve [12]. This Connes time τ has no analog in the commutative case, and we employ it here to describe evolution in trace dynamics. Thus we could define the action for gravity, in trace dynamics, as

$$S_{GTD} = \kappa \frac{c}{L_P} \int d\tau \; Tr[\chi(L_P^2 D^2)] \tag{23}$$

Note that S_{GTD} has the correct dimensions, that of action.

The above action defines the gravity part of the action for an 'atom of space-time-matter'. Next, we would like to derive the Lagrange equations for this trace action. For this we need to figure out what the configuration variables q are. In the presence of a manifold, those variables would simply be the metric. But we no longer have that possibility here. We notice though that the operator D is like momentum, since it has dimensions of inverse length. D^2 is like kinetic energy, so its trace is a good candidate Lagrangian. Therefore, we define a new bosonic operator q_B, having the dimension of length, and we define a velocity $dq_B/d\tau$, which is defined to be related to the Dirac operator D by the following new relation

$$D \equiv \frac{1}{Lc} \frac{dq_B}{d\tau} \tag{24}$$

where L is a length scale associated with the space-time atom. The action for the atom can now be written as action for gravity, in trace dynamics, as

$$S_{GTD} = \kappa \frac{c}{L_P} \int d\tau \; Tr[\chi(L_P^2 \; \dot{q}_B^2 / L^2 c^2)] \tag{25}$$

where the time derivative in $\dot{q}_B$ now indicates derivative with respect to Connes time. This clearly is the trace action for a free 'operator' particle where q_B represents the to-be-gravitational degree of freedom in non-commutative geometry. Only when a background manifold is available, do the q_B-s get related to the metric. We can now vary the action of the free particle with respect to the position operator q_B, and by taking the trace derivative in the Lagrange equations of motion it is easy to conclude that the equation of motion is $\ddot{q}_B = 0$. Thus the velocity is constant, and the Dirac operator is proportional to a constant matrix, and we can write the solution to the equations

of motion as an eigenvalue equation

$$D\psi = \frac{1}{L}\psi \tag{26}$$

where the state vector depends on Connes time τ, and on the gravitational degree of freedom q_B. When a manifold is available, then on scales below Planck length this equation is equivalent to Einstein equations, by virtue of the expansion (22). Note that on a manifold, this Dirac operator is same as the Dirac operator on a curved spacetime.

Next, we would like to have an action for fermions in our theory, on the same footing as the above action for an 'atom' of space-time. Thus, unlike the earlier non-commutative geometry based approaches to the standard model [18], where the spectral action including fermions takes the form $\{(Tr[D_B]^2) + \text{Fermionic Action}\}$, we would like to have a trace dynamics action of the form $Tr[D_B + D_F]^2$, where D_F is a newly introduced (non-self-adjoint) operator which represents fermions. In the emergent classical limit, this [symbolically] takes the form $(D_B^2 + D_B D_F + D_F^2)$ where the first term becomes the Einstein-Hilbert action, the second term becomes the Dirac action for a fermion (leading to the Dirac equation), and the last term the Higgs boson, which essentially comes for free in this construction.

Bringing fermions inside the square is challenging, but it can be done [7]. We define an odd-grade (hence fermionic) Grassmann matrix $\dot{q}_F$ to represent fermions, and define the 'fermionic Dirac operator' $D_F \equiv (1/Lc)\ dq_F/d\tau$. We define an 'atom' of space-time-matter, i.e. an aikyon, as $q \equiv q_B + q_F$. This is nothing but the splitting of a general Grassmann matrix into its even-grade (i.e. bosonic) and odd-grade (i.e. fermionic) part. One can construct an action principle for the aikyon in trace dynamics provided one gives up the squared Dirac operator in favour of a bilinear form! We introduce two constant fermionic (odd) Grassmann numbers β_1 and β_2 which must not be equal to each other. The action principle for an aikyon takes the form

$$S \sim \int d\tau\ Tr\left\{\frac{L_P^2}{L^2}[\dot{q}_B + \beta_1\dot{q}_F^\dagger] \times [\dot{q}_B + \beta_2\dot{q}_F]\right\} \tag{27}$$

It is not possible to make a consistent theory if $\beta_1 = \beta_2$. This immediately leads us to think of the aikyon as a 2-D object evolving in Connes time. In all likelihood, the aikyon is the same object as the closed string of string theory. However, at the Planck scale the dynamics of the aikyon is described by the laws of trace dynamics, not by the laws of quantum field theory. Hence our theory is different from string theory, even though the aikyon is likely the closed string. We would like to

suggest that string dynamics should be trace dynamics based on a (non-self-adjoint) Lagrangian. This trace (matrix) dynamics is not to be quantised. Rather, quantum (field) theory is emergent from it.

The transition from classical general relativity [gravitation coupled to relativistic point particles] to a pre-quantum, pre-space-time matrix dynamics is the most crucial step in the development of the Aikyon Theory. Hence we now elaborate on this transition carefully. The philosophy of trace dynamics is to start from a classical Lagrangian dynamics, and raise the c-number configuration and momentum variables to the status of matrices. These matrices are defined such that their eigenvalues are the original c-number configuration and momentum variables themselves. The new Lagrangian is defined as the trace of the matrix polynomial which results when the configuration variables and velocities in the original Lagrangian are replaced by matrices. The resulting Lagrangian matrix dynamics is trace dynamics - a pre-quantum dynamics - from which quantum theory is emergent. The trace Lagrangian must have an additional feature arising naturally: the corresponding Hamiltonian must not be self-adjoint in general. When the anti-self-adjoint part is small and ignorable, the emergent low-energy theory is quantum [field] theory. Under suitable circumstances [adequate entanglement amongst degrees of freedom] the anti-self-adjoint part becomes significant, and spontaneous localisation results. In this process, the trace Lagrangian is mapped to one of its eigenvalues, thereby converting the trace to the original c-number self-adjoint Lagrangian. The role of the anti-self-adjoint part is to provide imaginary stochastic fluctuations which, in the nature of objective collapse models, enable the quantum-to-classical transition. In this way, quantum theory, as well as classical dynamics, are both recovered from the pre-quantum trace dynamics, as suitable low energy emergent approximations.

In trace dynamics, space-time is Minkowski flat. We would like to incorporate gravity, by raising classical space-time and its geometry to matrix status. For this we must first identify a suitable way to express the classical theory. Let us begin by schematically writing down the action for classical general relativity:

$$S_{GR} \sim \int d^4x\, \sqrt{g}\, \frac{R}{L_P^2} + \sum_i m_i \int ds \tag{28}$$

[It is worth noting the remarkable fact that, in spite of being second order, the Einstein equations are linear in the source mass, and not quadratic, unlike the Klein-Gordon equation. This is an indicator that the mass term arises as a cross-term when a square is opened up.] We now employ the Dirac operator on this classical manifold, to express the gravity part of the action in terms of

the eigenvalues λ_i of the Dirac operator:

$$Tr\ [L_P^2\ D_B^2] \sim \int d^4x\sqrt{g}\ \frac{R}{L_P^2} + \mathcal{O}(L_P^0) \sim L_P^2 \sum_i \lambda_i^2 \tag{29}$$

so that the full action is

$$S_{GR} \sim L_P^2 \sum_i \lambda_i^2\ +\ \sum_i m_i \int d^4x\sqrt{g}\delta(\mathbf{x} - \mathbf{x_0}(t)) \tag{30}$$

To transit to the pre-space-time theory, *each* of the eigenvalues λ_i is now raised to the status of a matrix D_{Bi}, this being the original Dirac operator itself, so that we now have as many copies of the Dirac operator as the number of its eigenvalues, and

$$\lambda_i \to D_{Bi} \equiv \frac{1}{L}\frac{dq_{Bi}}{d\tau} \tag{31}$$

Here, q_{Bi} is the configuration variable, now a matrix/operator, which defines the i-th atom of space-time. Connes time τ has been brought in, because we are now in the domain of non-commutative geometry: the matrices q_{Bi} do not commute with each other. Classical space-time points have been lost, and we have a pre-quantum, pre-space-time dynamics. We refer to this as generalised trace dynamics [GTD]. The action for the i-th space-time atom will be $\int d\tau\ Tr[D_{Bi}^2]$, and the GTD action for the gravitation part of the theory will hence be

$$S_{GTD} \sim L_P^2 \sum_i \int d\tau\ Tr[D_{Bi}^2] \tag{32}$$

These are the atoms of space-time, from which space-time is emergent, after critical entanglement leads to spontaneous localisation. However, in our theory, matter in the form of fermions [coupled to gravity] is absolutely essential for the emergence of space-time [no matter, no space-time]. The points of the space-time manifold are defined by the c-number positions that fermions acquire after spontaneous localisation; these positions being the eigenvalues of the fermionic Dirac operator $D_F \equiv (1/Lc)\ dq_F/d\tau$ introduced above. We can define this fermionic Dirac operator more precisely, by first *assuming* that there are as many fermions as the number of eigenvalues of the Dirac operator D_B, and then by choosing the i-th operator $\dot{q}_{Fi}$ such that the eigenvalues of $\dot{q}_{Bi}\dot{q}_{Fi}$ [upon inspection of the matter action] are proportional to $m_i\delta(\mathbf{x} - \mathbf{x_0}(\mathbf{t}))$. Thus, the GTD action for gravity and

fermions can now be schematically written as

$$S_{GTD} \sim \sum_i S_i \sim \sum_i \int d\tau\, \left(Tr[\dot{q}_{Bi}^2] + Tr[\dot{q}_{Bi}\dot{q}_{Fi}] \right) \tag{33}$$

This action paves the way for introducing the aikyon - an atom of space-time-matter. Spontaneous localisation sends this trace Lagrangian above to the one shown in Eqn. (30) [each trace goes to an eigenvalue, and the Connes time integral stays as such]. Then from (30) we can re-construct the general relativity action (29). We emphasise that the action (33) is henceforth to be taken as the first-principles action, and no longer dependent on the general relativity action. All we need to know is that the eigenvalues are those of the Dirac operator D_B and of the fermionic Dirac operator D_F, which are in turn related to q_{Bi} and q_{Fi}. Space-time arises from 'collapse of the wave-function' [7, 19].

Taking clue from this form of the action above, we propose the following fundamental form for the action of an aikyon - an 'atom' of space-time-matter, thereby bringing the fermions 'inside the square':

$$\frac{S}{\hbar} = \int \frac{d\tau}{t_{Pl}}\, Tr \left\{ \frac{L_P^2}{L^2} \left[\dot{q}_B^\dagger + \frac{L_P^2}{L^2}\beta_1 \dot{q}_F^\dagger \right] \times \left[\dot{q}_B + \frac{L_P^2}{L^2}\beta_2 \dot{q}_F \right] \right\} \tag{34}$$

This is the action for the i-th aikyon, and the total action for the universe is the sum over all aikyons, of such individual terms. For most part of this paper, we will work with the action of only one aikyon. It is implicit that the total number of space-time dimensions is $4+1 = 3+1+1$, with the fifth dimension being Connes time. Soon however, we will conclude that the physical space must be doubled to have eight octonionic dimensions, making this a 8 + 1-d theory. With this doubling of spatial dimensions, the above action will be found to describe not just gravity, but also the weak interaction! If we think of the aikyon itself as a 2-d object [because of the two constants β_1 and β_2] then we have a 8+1+2 = 11-d theory. It is also known that an octonionic spacetime is equivalent to a 10D Minkowski spacetime.

The Lagrangian and action are not restricted to be self-adjoint. A dot denotes derivative with respect to Connes time. By varying this action w.r.t. q_B and q_F one gets a pair of coupled equations of motion, which can be solved to find the evolution of q_B and q_F. The respective momenta p_B and p_F are constants of motion, and the expression for p_B can be written as an eigenvalue equation for

the modified Dirac operator $D \equiv D_B + D_F$:

$$[D_B + D_F]\,\psi = \frac{1}{L}\left(1 + i\frac{L_P^2}{L^2}\right)\psi \tag{35}$$

where

$$D_B \equiv \frac{1}{Lc}\,\frac{dq_B}{d\tau}; \qquad D_F \equiv \frac{L_P^2}{L^2}\frac{\beta_1+\beta_2}{2Lc}\,\frac{dq_F}{d\tau} \tag{36}$$

D_B is defined such that in the commutative c-number limit where space-time emerges, it becomes the standard Dirac operator on a Riemannian manifold. D_F is defined such that upon spontaneous localisation, it gives rise to the classical action for a relativistic point particle.

In summary, we have followed in the footsteps of trace dynamics to generalise the pre-quantum theory to a pre-quantum pre-space-time theory. From this theory, quantum theory and classical gravitation are emergent as low-energy approximations.

VI. GENERALIZED TRACE DYNAMICS: A POSSIBLE LAGRANGIAN FOR GRAVITY, YANG-MILLS FIELDS, AND FERMIONS

Of course a limitation of the above dynamics is that it considers only gravity and Dirac fermions (albeit, unified in the aikyon concept in the Planck scale matrix dynamics). In the present section we recall how to include Yang-Mills gauge fields in our approach. For achieving this goal, we are guided by a few principles. Firstly, we would not like to lose out on the aikyon concept, which unifies a particle with its gravitation, by expressing them as $q = q_B + q_F$. As we have seen, what appears in the above action are not q_B and q_F themselves, but their time derivatives. These time derivatives are respectively identified with gravity and with the source of gravity, namely the material particles. The structure is symbolically of the form:

$$D^2 \equiv [D_B + D_F]^2 \sim D_B^2 + D_B\;D_F + D_F^2 \tag{37}$$

The first term on the right, i.e. D_B^2, leads to the gravity part of the Einstein-Hilbert action, whereas the second term, i.e. the cross-term $D_B\;D_F$, gives the relativistic point particle as the matter source. The last term, D_F^2, is a higher order term in the L_P^2 expansion, and is related to the Higgs boson. Einstein equations have the remarkable property that they are linear in the source mass, despite being second order equations. This makes them unlike the Klein-Gordon equation

which is second order, and also quadratic in the source mass. In this respect Einstein's equations are closer to the Dirac equation which is linear in the source mass. This linear dependence on the mass in Einstein equations is easily understood in our theory, because they originate from the first order equation (35) and the mass term arises from the term linear in D_F in the above expansion of D^2.

We also know that when spontaneous localisation localises the fermionic part of an aikyon to a specific position, the associated space-time manifold, metric, and gravitational field, emerge concurrently with the localisation. The same feature will have to be true for the Yang-Mills field produced by its associated charge and its current: the localised current must emerge concurrently with its associated gauge field, and these two must emerge concurrently with the localised mass and associated gravitation of the aikyon.

These remarks guide us as to how Yang-Mills fields can be included. We know that in the Dirac equation they enter as an 'internal' connection in the form: $D_B \longrightarrow D_B + \alpha A$. Since in our matrix dynamics D_B is identified with the velocity $dq_B/d\tau$, we propose to identify the (self-adjoint) gauge-field potential operator A with q_B. This way we make the gauge-field and gravitation respectively the position and velocity aspects of the aikyon. Similarly, since D_F, which gives rise to the source mass term, is identified with the velocity $dq_F/d\tau$, we propose to identify the current j (which is the source of the gauge field) with q_F. Thus, the new squared Dirac operator will be of the form

$$D_{new}^2 \equiv [D_B + \alpha A + D_F + j]^2 \sim D_B^2 + \alpha^2 A^2 + \alpha D_B A + D_B D_F + D_B j + \alpha A D_F + \alpha A j + D_F^2 + j^2 + D_F j \tag{38}$$

This includes the action term for the gauge fields, symbolically written as $\alpha^2 A^2$, and their source term $D_B j$ (which is linear in the current) apart from the gravitation terms we already have earlier, and a few new terms. Hence, by including the gauge-fields and their charges in the action for the aikyon, we will derive Einstein equations with gauge fields and material particles as source. We will also derive quantum field theory for these gauge fields coupled to Dirac fermions.

In arriving at Einstein equations coupled to gauge-fields, after spontaneous localisation from this matrix dynamics, we will make use of the following result from geometry, for the heat kernel expansion of the bosonic part $D_{Bnew} \equiv D_B + \alpha A$ of D_{new}:

$$Tr\ [L_P^2\ D_{Bnew}^2] \propto L_P^{-2} \int d^4x \sqrt{g} \left(R + L_P^2\, \alpha^2 F_{\mu\nu}^i F_i^{\mu\nu}\right) + \mathcal{O}(L_P^2) \tag{39}$$

This generalises the corresponding result above, when there is only gravity, represented by D_B^2,

but no gauge-field A. This result is motivated by the careful and pioneering work of Chamseddine and Connes on the heat kernel expansion of the modified squared Dirac operator $P = D^2_{Bnew}$ when a correction due to the presence of a Yang-Mills field is included. In this context, we note the following from Section II [The spectral action principle applied to the Einstein-Yang-Mills system] of their paper [20] where they discuss the spectral action when Yang-Mills fields are included. As they say, it is possible to introduce a mass scale m_0 and consider χ to be a function of the dimensionless variable $\chi\left(\frac{P}{m_0^2}\right)$. In this case terms in a heat kernel expansion coming from $a_n(P)$, $n > 4$ are suppressed by the powers of $\frac{1}{m_0^2}$, and one gets for $Tr\ \chi\left(\frac{P}{m_0^2}\right)$ the following important result:

$$
\begin{aligned}
I_b = \frac{N}{48\pi^2}\bigg[& 12m_0^4 f_0 \int d^4x\sqrt{g} + m_0^2 f_2 \int d^4x\sqrt{g}R \\
& + f_4 \int d^4x\sqrt{g}\bigg[-\frac{3}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{20}R^*R^* + \frac{1}{10}R;_\mu{}^\mu \\
& + \frac{g^2}{N}F^i_{\mu\nu}F^{\mu\nu i}\bigg] + \mathcal{O}\left(\frac{1}{m_0^2}\right)\bigg]
\end{aligned}
\tag{40}
$$

where

- $\frac{Nm_0^2 f_2}{48\pi^2}\int d^4x\sqrt{g}R$ term is the Einstein-Hilbert action
- $\frac{Nm_0^4 f_0}{4\pi^2}\int d^4x\sqrt{g}$ term is responsible for the cosmological constant
- $\frac{f_4 g^2}{48\pi^2}\int d^4x\sqrt{g}F^i_{\mu\nu}F^{\mu\nu i}$ term is the Yang-Mills action
- $-\frac{Nf_4}{320\pi^2}\int d^4x\sqrt{g}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ term would be responsible for the Conformal gravity
- $\frac{11Nf_4}{960\pi^2}\int d^4x\sqrt{g}R^*R^*$ term would be responsible for the Gauss-Bonnet gravity"

This is the expansion of the squared Dirac operator when gauge fields are included alongside gravity. In our case, we set the scale m_0 to be the inverse of Planck length. Also, for now, we do not take into account the volume term, and conformal gravity, and Gauss-Bonnet gravity in our present work. Note though that their analysis is classical; whereas we will employ it to construct a matrix dynamics from which quantum theory emerges. Furthermore, we will return to consideration of the significant conformal gravity term when we discuss the possible connection of our work with MOND.

When spontaneous localisation results in the localisation of the fermionic part of an aikyon, it simultaneously gives rise to the space-time manifold as well as the gravitational field. Now, if

the fermion has a charge, such as an electric charge, the associated electromagnetic field must also appear along with gravitation. That is the goal accomplished by including Yang-Mills fields. From this point of view, it is difficult to come to the conclusion that the space-time manifold and its associated curvature are in any sense more fundamental than the gauge fields which supposedly 'live' 'on' space-time. The gauge-fields of a fermion, and their associated charge α, are as fundamental as space-time-gravitation of the fermion, with its associated length L.

The Lagrangian (34) can thus be extended to include Yang-Mills fields q_B [21] via the following generalisation

$$\frac{S}{\hbar} = \int \frac{d\tau}{t_{Pl}}\, Tr\Bigg\{ \frac{L_P^2}{L^2}\left[\left(\dot{q}_B^\dagger + i\frac{\alpha}{L}q_B^\dagger\right) + \frac{L_P^2}{L^2}\beta_1\left(\dot{q}_F^\dagger + i\frac{\alpha}{L}q_F^\dagger\right)\right]\times\left[\left(\dot{q}_B + i\frac{\alpha}{L}q_B^\dagger\right) + \frac{L_P^2}{L^2}\beta_2\left(\dot{q}_F + i\frac{\alpha}{L}q_F\right)\right]\Bigg\} \tag{41}$$

which can also be written as

$$\frac{S}{\hbar} = \int \frac{d\tau}{\tau_{Pl}}\, Tr\Bigg\{ \frac{L_P^2}{L^2}\left[\left(\dot{q}_B^\dagger + \frac{L_P^2}{L^2}\beta_1\dot{q}_F^\dagger\right) + i\frac{\alpha}{L}\left(q_B^\dagger + \frac{L_P^2}{L^2}\beta_1 q_F^\dagger\right)\right]\times\left[\left(\dot{q}_B + \frac{L_P^2}{L^2}\beta_2\dot{q}_F\right) + i\frac{\alpha}{L}\left(q_B + \frac{L_P^2}{L^2}\beta_2 q_F\right)\right]\Bigg\} \tag{42}$$

Here, q_B describes Yang-Mills fields, q_F as well the earlier $\dot{q}_F$ describe fermions. α is the dimensionless coupling constant which describes the coupling of Yang-Mills fields to fermions. This is the form of the Lagrangian for an aikyon. It is a unified description of gravitation, Yang-Mills fields, and fermions, and is hence a candidate for investigating the unification of gravitation with the standard model of particle physics. This approach to unification has been studied in the present programme, wherein this Lagrangian in pre-spacetime, pre-quantum theory has been explored. Very significantly, we are compelled to introduce an underlying non-commutative spacetime - an 8D octonionic spacetime, which is equivalently a 10D Minkowski spacetime. That makes all the difference, by bringing in the standard model of particle physics, and by opening up an unforeseen avenue for unification of gravitation with the three other fundamental forces.

VII. AN OCTONIONIC SPACE-TIME FOR THE LAGRANGIAN IN GENERALIZED TRACE DYNAMICS: TOWARDS QUANTUM GRAVITY AND UNIFICATION

Generalised trace dynamics, being a Lagrangian dynamics, must offer a canonical definition of spin angular momentum. This definition should in turn agree with the definition of spin as coming from Poincare symmetry in 4D Minkowski space-time. In a recent paper [22] we have shown, in the following manner, that quantum spin is defined as the canonical angular momentum corresponding

to the time variation of a certain angle. We have seen above in the Lagrangian (42) that the Yang-Mills field is introduced with an i factor (only then does one get finite solutions to the equations of motion [21]). This permits us to think of $\dot{q}_B$ and q_B as if they are the real and 'imaginary' components of $\dot{q}_B + iq_B$. Writing this sum in the equivalent form $R_B \exp i\theta_B$ and substituting it in the Lagrangian permits us to define spin angular momentum as proportional to $\dot{\theta}_B$; spin results from the time rate of change of the phase that relates the Lorentz field $\dot{q}_B$ with Yang-Mills field q_B via a rotation. It is shown heuristically that this definition gives the correct conventional interpretation of spin - more work remains to be done to make this connection precise. Also bosonic Grassmann matrices are shown to have integral spin, and fermionic matrices are shown to have half-integral spin, in the emergent theory. Instead of deriving statistics from spin, we have derived spin from statistics, thus providing a simple proof of the spin-statistics connection.

In constructing this definition of spin, we noticed something very curious. $\dot{q}_B$ lies along four real directions [space-time]. It is as if q_B does not lie in space-time, but along an orthogonal set of four imaginary directions (it is natural to expect four components of q_B, if $\dot{q}_B$ has four). That is a total of eight directions. This has far-reaching implications! Firstly, it appears that the right place for this Lagrangian is phase space, not space-time: we have 'position' q_B and velocity $\dot{q}_B$. Both have their own independent interpretation: just like the (q, p) pair in phase space. Secondly, ours is a non-commutative space, and eight non-commuting directions immediately suggests the octonions! There is a rich history of relating octonions to the standard model, and those findings [23–40] open the gateway for relating our Lagrangian to division algebras, allowing us to unify the Lorentz symmetry with the standard model, and propose a unified theory. Thirdly, it is known that eleven dimensional string theory [M-theory] is akin to having ten space-time dimensions, plus the eleventh dimension possibly serving as the second time [Connes time ?]. It is also known those ten space-time dimensions are equivalent to eight octonionic dimensions [41]. So it could well be that the self-adjoint part of the Hamiltonian of our theory, at energies below Planck scale, describes the same theory as string theory. However, we do not need dimensional compactification or Calabi-Yau manifolds. Dimensional reduction is naturally achieved for classical systems via the dynamical process of spontaneous localisation. Quantum systems continue to remain and evolve in eight dimensional octonionic space. An aikyon can be thought of as evolving in this 8-D non-commutative phase space in Connes time. The algebra automorphisms of the octonions take the place of gauge transformations [internal symmetries]. But they also take the place of space-time diffeomorphisms. There is no longer a Diff M at the Planck scale. Instead, the algebra automorphisms take the place of Diff M and of internal gauge transformations, and unify them

into one concept. General covariance and gauge invariance are unified, and the group formed by the automorphisms of octonions is the smallest of the exceptional Lie groups, G_2. This then will be the symmetry group for one generation of fermions unified with the gauge interactions and with Lorentz symmetry [more precisely, it is the group obtained from the complex octonions, this being $SL(2,\mathbb{O}) \sim SO(1,9)$]. An octonionic space-time is defined by the eight direction vectors of an octonion:

$$O = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7 \tag{43}$$

where $e_0 = 1$, and the other e_i are seven imaginary unit directions, each of which square to minus one, and obey the Fano plane multiplication rules. A dynamical variable, this being a matrix q as in trace dynamics, will have eight component matrices; thus

$$q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 + q_4 e_4 + q_5 e_5 + q_6 e_6 + q_7 e_7 \tag{44}$$

In our theory, $\dot{q}_B$ will have four octonionic directional components, including the real direction, so that $\dot{q}_B$ will be the quaternion part of the octonion. q_B will occupy the other four octonion directions. Together, the q_B and $\dot{q}_B$ will have eight directions and components - they are a set of eight complex-numbered bosonic Grassmann matrices over the eight octonionic directions. Similarly, the $\dot{q}_F$ and q_F, the fermions, together have eight octonionic components. Our quadratic Lagrangian has the structure of complex octonions acting on themselves; hence the relevance of division algebra studies for our theory.

With this background of earlier work, we are now ready to relate the Lagrangian of the theory to the standard model, and to division algebras, sedenions, and the exceptional Jordan algebra. Most importantly, in our theory we were in need of a physical non-commutative space in which the aikyon lives, and we have found one in the octonions. On the other hand, the profound studies relating division algebras to the standard model are badly in need of a Lagrangian! Their story is incomplete if there are only symmetries but no Lagrangian whose symmetries those are. The Lagrangian we have constructed in trace dynamics turns out to be perfect for relating to division algebras. There is no room for manoeuvre, no fine-tuning and no free parameters at all. If the predictions of this Lagrangian disagree with known physics [our theory is eminently falsifiable] then this theory will be totally ruled out. It cannot be saved by making alterations or adjustments. Division algebras cannot be applied to the conventional standard model Lagrangian, because that Lagrangian resides

in space-time and is generally covariant. It is not in need of algebra automorphisms. These algebra automorphisms become key at the Planck scale. And even more importantly, the trace dynamics Lagrangian does not have to be quantised. It already describes a dynamics from which quantum theory is emergent. And the trace Lagrangian is invariant under global unitary transformations constructed from the generators of these automorphisms.

The relevance of algebras to particle physics goes back to the 1934 paper by Jordan, von Neumann and Wigner [42], and Albert [43] who discovered the significance of the exceptional Jordan algebra: the algebra of 3×3 Hermitean matrices with octonionic entries. There was then some lull until in the 1970s when Gunaydin and Gursey [44] showed that important properties of quarks and leptons can be inferred from the algebra of octonions acting onto themselves. This was followed by important studies by various researchers including Dixon and Baez. From our point of view, the fast-paced developments during the last five years or so are extremely significant, and are already pointing to the correct symmetry group for unification. In arriving at the findings of the present paper, we have been inspired amongst others by the recent research of Furey [25–27], of Stoica [45], of Gillard and Gresnigt [46, 47], of Dubois-Violette and Todorov [32], Dray and Manogue [48], and Boyle [49]. In arriving at the unified theory proposed in this paper, we build heavily on their work, and explain it in the context of our Lagrangian, and show how our Lagrangian fits several of their findings.

In her 2016 Ph. D. thesis [25], Furey explains, building on the work of earlier researchers, the significance of minimal left ideals constructed from the algebra of complex quaternions acting on themselves. The complex quaternions give a faithful representation of the Clifford algebra $C\ell(2)$. Spinors are minimal left ideals of Clifford algebras. The left and right handed Weyl spinors are minimal left ideals made from the action of $C\ell(2)$ generators on the idempotent. These Weyl spinors transform correctly under $SL(2, C)$, which gives the Lorentz algebra and is also the group of automorphisms of the complex quaternions. In our Lagrangian (45) below, the symmetry group leaving the first term invariant contains this automorphism group. Hence, as per the aikyon concept, this term includes the Lorentz 'interaction', which is mediated by the two Lorentz bosons.

In an analogous manner Furey then shows that the complex octonions can be used to deduce the Clifford algebra $C\ell(6)$ which has six generators. These generate an eight dimensional basis. Now, it turns out that the generators have a $U(3) \sim SU(3) \times U(1)/\mathbb{Z}_3$ symmetry. The $SU(3)$ is an element preserving subgroup of G_2, the automorphism group of the octonions, and the $U(1)$ is a number operator made from these generators. Minimal left ideals are made by left multiplying $C\ell(6)$ on the idempotent, giving rise to the said 8-D basis. Now, the $U(3)$ symmetry imposes a definite

discrete structure on this basis, including charge quantizaton, compelling us to identify it with one generation of quarks and leptons. Thus the algebra of complex octonions describes the eight fermions of one generation in the standard model. The unbroken $SU(3)_c \times U(1)_{em}$ symmetry of the standard model can be related to a division algebra. Three of the eight terms in our Lagrangian relate to this $U(3)$ symmetry.

But what about the $SU(2)$ weak symmetry, and what about the unbroken electro-weak symmetry? How to relate them to a division algebra? In a 2018 paper [27] Furey employs four of the electro-colour generators to make an $SU(2)$ symmetry from the Clifford algebra $C\ell(4)$. This symmetry acts correctly on the leptons, as expected from the standard model. However, it would appear that this cannot be a fundamental construct, because the generators have come from the electro-colour algebra. And the octonions can give only one representation of $C\ell(6)$, which is already used up. So the complex octonions by themselves seem like an unlikely candidate for explaining the weak symmetry. In another elegant 2018 paper, Stoica showed [45] that two copies of $C\ell(6)$, which he proposes to identify, describe correctly the symmetries of one generation of the eight quarks and leptons, *including* the Lorentz symmetry. His construction is not concerned with division algebras. Again, four of the six generators of the weak-Lorentz sector can in principle be constructed from the electro-colour sector. This is how things stood until our recent work [50].

A careful analysis of these two papers led us to the propose the Lorentz-weak unification. In order to arrive at this conclusion, we have to look carefully at the two maximal sub-groups of G_2 [51]. One of them is the element preserving group of the octonions; it is the $SU(3)$. The other maximal sub-group is the stabiliser group of the quaternions in the octonions. It is $SU(2) \times SU(2)/\mathbb{Z}_2$, and does not seem to have gotten the attention it deserves. Although Todorov and Drenska [33] do prove its existence. It has a sub-group $SO(3)$ which is the group of automorphisms of the quaternions. The group extension of this $SO(3)$ is an $SU(2)$ which happens to be the element preserving group of the quaternions. Moreover, the two maximal subgroups have a $U(2)$ intersection, of which the said $SU(2)$ is a normal subgroup. Todorov and Dubois-Violette [52] note that this $U(2)$ is precisely the Weinberg-Salam electro-weak model! Taking clue from this group intersection, we propose the Lorentz-weak symmetry [using complex octonions, so that one of the $SU(2)$ is replaced by $SL(2,C)$]: prior to the electro-weak symmetry breaking, the Lorentz symmetry is unified with the weak symmetry, and hence with electro-weak. The symmetry group for the Lorentz-weak symmetry is obtained from 'complexifying' the stabiliser group of the quaternions inside the octonions, and is $SL(2,C) \times SU(2)$. When spontaneous localisation separates spacetime and hence the Lorentz symmetry, the electro-weak becomes part of the internal symmetries, along with $SU(3)_c$. Three

terms in our Lagrangian describe the Lorentz-weak symmetry and lead us to predict the Lorentz bosons. Another two terms describe the bosons, and the remaining two describe what are likely the Higgs bosons. Hence the Lagrangian describes the standard model bosons, and one fermion generation. The $U(2)$ intersection of the two sub-groups, wherein the weak symmetry lies, explains why the generators of the weak symmetry can be constructed from those of $SU(3)_c$. And, to describe the Lorentz-weak symmetry using a Clifford algebra, we must indeed look beyond the complex octonions and beyond the four division algebras, because the complex octonions can give only one representation of $Cl(6)$.

Consequently, the subsequent analysis leads to a proposal for unification of interactions. It turns out that if we retain in the Lagrangian the total time derivative terms which were dropped to arrive at the above-mentioned one generation Lagrangian, something remarkable happens. The Lagrangian now has the quadratic form of complex sedenions acting on themselves. Following the recent work of Gillard and Gresnigt [46], we show that the full Lagrangian describes the symmetries of three fermion generations, including the unification with the Lorentz interaction. This then suggests a unification of the four interactions of the known elementary particles. The symmetry group is made of three [intersecting] copies of G_2, which are embedded in the exceptional Lie group F_4. This discussion is then extended to E_6, the unification group of the unified theory. The self-adjoint part of our three-generation Lagrangian is related to the exceptional Jordan algebra $J_3(\mathbb{O})$, whose automorphism group is again the same F_4. The (cubic) characteristic equation of this algebra determines the mass ratios of three generations of elementary particles, and the low energy fine structure constant, and perhaps (?) values of the other free standard model parameters as well.

The next sub-section describes in some detail the trace dynamics Lagrangian and its connection with the standard model.

A. Trace dynamics, division algebras, and the standard model

The following Lagrangian with eight terms arises from opening up the form shown in (42) [21]. That opening up gives rise to sixteen terms, eight of which are total time derivatives and are discarded for now. We will return to these discarded terms later in the paper - they will be needed

for describing three fermion generations.

$$\frac{S}{\hbar} = \frac{1}{2}\int \frac{d\tau}{\tau_{Pl}}\; Tr\Big[\frac{L_P^2}{L^2}\Big\{\Big(\dot{q}_B^\dagger \dot{q}_B + \frac{L_P^2}{L^2}\dot{q}_B^\dagger \beta_2 \dot{q}_F + \frac{L_P^2}{L^2}\beta_1 \dot{q}_F^\dagger \dot{q}_B + \frac{L_P^4}{L^4}\beta_1 \dot{q}_F^\dagger \beta_2 \dot{q}_F\Big) \\ -\frac{\alpha^2}{L^2}\Big(q_B^\dagger q_B + \frac{L_P^2}{L^2} q_B^\dagger \beta_2 q_F + \frac{L_P^2}{L^2}\beta_1 q_F^\dagger q_B + \frac{L_P^4}{L^4}\beta_1 q_F^\dagger \beta_2 q_F\Big)\Big\}\Big] \tag{45}$$

This Lagrangian is not self-adjoint [because β_1 and β_2 are unequal] and this plays a crucial role when we use division algebras to relate to the standard model. The presence of the anti-self-adjoint part is also key for inducing spontaneous localisation and emergence of classical limit.

By defining the dynamical variables $q_1^\dagger = q_B^\dagger + \beta_1 q_F^\dagger$ and $q_2 = q_B + \beta_2 q_F$ this trace Lagrangian can be elegantly written as [53]

$$Tr\mathcal{L} = \frac{1}{2} a_1 a_0 \; Tr\left[\dot{q}_1^\dagger \dot{q}_2 - \frac{\alpha^2 c^2}{L^2} q_1^\dagger q_2\right] \tag{46}$$

where $S \equiv \int d\tau \; Tr\mathcal{L}$ and $a_0 \equiv L_P^2/L^2$ and $a_1 \equiv \hbar/cL_P$. This is the fundamental Lagrangian which describes an aikyon $q = q_B + q_F$ in terms of $q_1^\dagger$ and q_2. The equations of motion follow from variation of this Lagrangian [trace derivatives] with respect to the matrix degrees of freedom $q_1^\dagger$ and q_2. These are the defining dynamical equations of the theory, which is *not* quantised. Rather, quantum field theory is shown to emerge as a statistical thermodynamics approximation to the underlying trace dynamics, the latter assumed to be operating at the Planck scale. The idea being that if we are not observing the Planckian dynamics, but only observing the dynamics at low energies, we coarse-grain the underlying theory over length / time scales much larger than Planck length / Planck time, and ask what the laws of the emergent dynamics are. These laws are those of quantum field theory. However, the Planck scale laws are those of trace dynamics, different from, and more general than, those of quantum field theory [6].

For ease of further reference, we will label the eight terms in the Lagrangian (45), starting from the left, as terms $T_1, T_2, T_3, .., T_8$. As we will see, when we describe this Lagrangian in octonionic space, the first three terms T_1, T_2 and T_3 describe Lorentz symmetry of 4-D non-commutative space-time, and weak isospin, and their interaction with eight fermions of one generation, and the $SU(2)$ symmetry of weak interactions. These behave like kinetic energy terms. The terms T_5, T_6, T_7, the one with the coupling constant α, describe strong interactions and electromagnetism, the eight gluons and the photon and their interactions with fermions, and $SU(3)_c \times U(1)_{em}$ symmetry. These are like potential energy terms. The terms T_4 and T_8 form the heart of the division algebra program, from our point of view, and explain how minimal left ideals made from action of complex

octonions onto themselves determine properties of quarks and leptons. These terms could also possibly describe the Higgs bosons, as composites of fermions. We can also pair the terms as (T_1, T_5), (T_2, T_6), (T_3, T_7) and then they can be thought as the [kinetic energy + potential energy] of four different 'particles': i.e. $(q_B, q_B^\dagger, q_F, q_F^\dagger)$. Similarly in (46) the potential energy terms describe the strong and electromagnetic interactions of the fermions, and the kinetic energy terms describe their gravitational and weak interactions.

B. An octonionic space for the Lagrangian

In our recent papers [22, 53] we have motivated why the space of quaternions and octonions is the appropriate setting for describing the dynamics of an aikyon using the above Lagrangian, and for relating this dynamics to the standard model. We now develop this construction in full detail. The aikyon evolves in an octonionic coordinate system in Connes time, and the various bosons and fermions of the standard model relate to different directions in this 8-D coordinate system. The Lagrangian above describes their dynamics and interactions.

An octonion has one real direction, which we denote as e_0, and seven imaginary directions, $(e_1, e_2, ..., e_7)$. The square of the real direction is unity: $e_0^2 = 1$, and the square of each imaginary direction is -1. We write the octonionic coordinate system as $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$. A quaternion has one real direction e_0 and three imaginary directions which we will label (e_1, e_2, e_4). The quaternionic product rule is

$$e_1 \times e_2 = -e_2 \times e_1 = e_4; \qquad e_2 \times e_4 = -e_4 \times e_2 = e_1, \qquad e_4 \times e_1 = -e_1 \times e_4 = e_2 \tag{47}$$

We will work with complex quaternions and complex octonions. The product rule for the octonionic directions is given by the Fano plane. For our ready reference, we display these products explicitly below, in Table I. The seven imaginary directions come in quaternionic triples, meaning that every triple obeys the quaternionic product rule, and along with unity forms a closed quaternionic sub-algebra of the octonion algebra. In our notation, the seven triples are: $(e_1, e_2, e_4), (e_3, e_4, e_6), (e_6, e_1, e_5), (e_5, e_2, e_3), (e_3, e_7, e_1), (e_5, e_7, e_4), (e_6, e_7, e_2)$. The 8x8 product table is quite remarkable, and deserves closer attention. We think of the table as made of 4x4 sub-tables, to which we give the self-explanatory names Top-left, Top-right, Bottom-left, Bottom-right. These four sub-tables have interesting properties. The Top-left forms the quaternionic sub-algebra of octonions which has the real direction; it is the only sub-algebra in the algebra of octonions

	e0	e1	e2	e4	e3	e5	e6	e7
e0	1	e1	e2	e4	e3	e5	e6	e7
e1	e1	-1	e4	-e2	e7	e6	-e5	-e3
e2	e2	-e4	-1	e1	e5	-e3	e7	-e6
e4	e4	e2	-e1	-1	-e6	e7	e3	-e5
e3	e3	-e7	-e5	e6	-1	e2	-e4	e1
e5	e5	-e6	e3	-e7	-e2	-1	e1	e4
e6	e6	e5	-e7	-e3	e4	-e1	-1	e2
e7	e7	e3	e6	e5	-e1	-e4	-e2	-1

FIG. 1. The multiplication table for two octonions. Elements in the first column on the left, left multiply elements in the top row. We follow the notation of [25]. (e_0, e_1, e_2, e_4) form a quaternion that emerges as space-time.

which has the real direction in it. It is made from the directions (e_0, e_1, e_2, e_4) and we assign the bosonic Grassmann matrix $\dot{q}_B$, which describes gravity and weak interactions in the above Lagrangian, these four directions. We are assuming that the directions can be treated as 'constants' which commute with the Grassmann elements. Thus $\dot{q}_B$ is a bosonic complex Grassmann matrix over the field of quaternionic numbers. These four directions will eventually be identified with emergent 4-D space-time; with the real direction being time coordinate and the other three being spatial directions. We will often refer to these four directions as the 'lower-half octonion plane', in contrast to the remaining four directions (e_3, e_5, e_6, e_7) which we will refer to as the 'upper half octonion plane'. The nomenclature is quite natural, from the viewpoint of our Lagrangian, as the lower plane has the character of a real line, and the upper plane the character of the imaginary line of a complex plane. If we look at the Top-right 4x4 plane, which comes from multiplying entries from the lower-half-plane and the upper-half-plane, it has all entries from only (e_3, e_5, e_6, e_7). It is

just like getting a pure imaginary number when we multiply a real number with a pure imaginary number. The same is true of the sub-table Bottom-left. We will assign the dynamical variable q_B, which describes Yang-Mills fields in our Lagrangian, the upper-half plane directions (e_3, e_5, e_6, e_7). As if they were the imaginary part of the unified force $(\dot{q}_B, q_B)$. In fact, in our earlier paper [21] the Yang-Mills fields were introduced precisely as the imaginary counterpart to gravity. The fact that this gels exactly with the octonion is encouraging. Together, $(\dot{q}_B, q_B)$ form an octonionic bosonic Grassmann matrix. The entries in the Bottom-right table are all from the lower-half plane (e_0, e_1, e_2, e_4), which is reasonable: they have come from squaring the top-right: square of an imaginary number is real. This is also what puts the square q_B^2 in the 'real / self-adjoint' part of the 8x8 matrix, as desired, even though q_B itself is pure imaginary. Thus the bosonic part lies entirely in the Top-left. The Top-left and Bottom-right sub-tables are 'real' and the Top-right and Bottom-left are 'imaginary'.

We will assume q_F to be four-dimensional, and $\dot{q}_F$ to be also four-dimensional. Together they form an octonionic fermionic Grassmann matrix, and their respective directions are chosen, for the purpose of the present discussion, in the following not-so-obvious manner, though the reason for this choice will become clear in the next sub-section. Similarly, $q_F^\dagger$ and $\dot{q}_F^\dagger$ form an octonionic Grassmann matrix with the shown directions. Thus we label the eight fermions as having the following eight linearly independent octonionic directions: $(V_\nu, V_{ad1}, V_{ad2}, V_{ad3}, V_{e+}, V_{u1}, V_{u2}, V_{u3})$ which stand respectively for the neutrino, the three anti-down quarks, the positron, and the three up quarks. The anti-particle directions are simply the complex conjugate of the corresponding particle direction [not the self-adjoint]. Thus, $(V_{a\nu}^*, V_{d1}^*, V_{d2}^*, V_{d3}^*, V_{e-}^*, V_{au1}^*, V_{au2}^*, V_{au3}^*)$ denote respectively the anti-neutrino, three down quarks, the electron and three anti-up quarks.

$$\begin{aligned}
\beta_1 \dot{q}_F^\dagger &= \Big(V_\nu, V_{ad1}, V_{ad2}, V_{ad3}\Big) \qquad [\text{Neutrino, Anti}-\text{down quarks}] \\
\beta_2 \dot{q}_F &= \Big(V_{e+}, V_{u1}, V_{u2}, V_{u3}\Big) \qquad [\text{Positron, Up quarks}] \\
\beta_1 q_F^\dagger &= \Big(V_{a\nu}^*, V_{d1}^*, V_{d2}^*, V_{d3}^*\Big) \qquad [\text{Anti}-\text{neutrino, Down quarks}] \\
\beta_2 q_F &= \Big(V_{e-}^*, V_{au1}^*, V_{au2}^*, V_{au3}^*\Big) \qquad [\text{Electron, Anti}-\text{up quarks}]
\end{aligned} \tag{48}$$

The names of the particles shown will be justified in the next sub-section. We note the peculiarity that the anti-particles are obtained by taking the complex conjugate, and not by taking the adjoint of the particle. Thus the Higgs is perhaps composite of several [particle, octonionic-conjugate-of-anti-particle] pairs. It is not clear to us what implication the presence of the octonionic conjugate

might have, in such constitution of the Higgs.

C. Using division algebras to relate the Lagrangian to the standard model

1. *The overall picture*

The aikyon lives in an 8D octonionic coordinate system, and the dynamics is described by complex-valued Grassmann matrices. These matrices have matrix-valued components in the 8D octonionic coordinate system. Thus, the bosonic matrix $\dot{q}_B$ introduced above and having components along the quaternion directions (e_0, e_1, e_2, e_4) can be written as $\dot{q}_B = \dot{q}_{Be0}\ e_0 + \dot{q}_{Be1}\ e_1 + \dot{q}_{Be2}\ e_2 + \dot{q}_{Be4}\ e_4$. The components along the other four directions are zero. Similarly, the bosonic matrix q_B has components along (e_3, e_5, e_6, e_7) and can be written as $q_B = q_{Be3}\ e_3 + q_{Be5}\ e_5 + q_{Be6}\ e_6 + q_{Be7}\ e_7$. Together, they form the octonionic-coordinate based bosonic Grassmann matrix $(\dot{q}_{Be0}\ e_0 + \dot{q}_{Be1}\ e_1 + \dot{q}_{Be2}\ e_2 + \dot{q}_{Be4}\ e_4 + q_{Be3}\ e_3 + q_{Be5}\ e_5 + q_{Be6}\ e_6 + q_{Be7}\ e_7)$. The components of these eight matrices are even-grade complex Grassmann numbers. A similar interpretation holds for the fermionic odd-grade matrices present in the Lagrangian above.

The automorphisms of the octonion algebra form the group G_2, which is the smallest of the exceptional Lie groups, and which has fourteen generators. From these generators, one can construct unitary transformations, and these unitaries, when they act on individual dynamical variables, leave the trace Lagrangian unchanged, as is also known in the theory of trace dynamics [because of the allowed cyclic permutations inside the trace]. To possibly see this in another way, we recall from our earlier work that the above Lagrangian (42) can be brought to the following form after a redefinition of variables [21]:

$$\mathcal{L} = Tr\left[\frac{L_p^2}{L^2}\left(\dot{\widetilde{Q}}_B^\dagger + \frac{L_p^2}{L^2}\beta_1\dot{\widetilde{Q}}_F^\dagger\right)\left(\dot{\widetilde{Q}}_B + \frac{L_p^2}{L^2}\beta_2\dot{\widetilde{Q}}_F\right)\right]$$

where

$$\dot{\widetilde{Q}}_B = \frac{1}{L}(i\alpha q_B + L\dot{q}_B); \qquad \dot{\widetilde{Q}}_F = \frac{1}{L}(i\alpha q_F + L\dot{q}_F); \tag{49}$$

This trace Lagrangian can also be usefully written as

$$\mathcal{L} = Tr\left[\frac{L_p^2}{L^2}\left\{\dot{\widetilde{Q}}_B^\dagger\dot{\widetilde{Q}}_B + \frac{L_p^2}{L^2}\left(\beta_1\dot{\widetilde{Q}}_F^\dagger\dot{\widetilde{Q}}_B + \dot{\widetilde{Q}}_B^\dagger\beta_2\dot{\widetilde{Q}}_F\right) + \frac{L_p^4}{L^4}\beta_1\dot{\widetilde{Q}}_F^\dagger\beta_2\dot{\widetilde{Q}}_F\right\}\right] \tag{50}$$

This makes the unification of interactions manifest. The first term is the bosonic part of the

Lagrangian, the second and third term are the action of the bosons on the fermions, and the fourth term, the fermionic kinetic energy, is central for us to make contact with division algebras, and possibly also represents the Higgs boson. Spontaneous localisation, when it happens, separates gravitation, and its action on the fermions, from the internal symmetries. The weak symmetry separates from space-time and gravity, and combines with electromagnetism to form the electroweak symmetry. We call the entity described by the above Lagrangian, an 'atom of space-time-matter' or an aikyon. The known elementary particles (quarks and leptons) and the gauge bosons, are special cases of the aikyon. Each of the four terms in this Lagrangian has the form of an octonion acting on itself, which helps understand why minimal left ideals of their algebra and the associated Clifford algebras are so important for understanding the standard model.

The trace Lagrangian above can be written even more compactly as

$$\mathcal{L} = Tr\left[\frac{L_p^2}{L^2}\dot{\tilde{Q}}_{1sed}^\dagger\, \dot{\tilde{Q}}_{2sed}\right] \tag{51}$$

where

$$\dot{\tilde{Q}}_{1sed}^\dagger = \dot{\tilde{Q}}_B^\dagger + \frac{L_p^2}{L^2}\beta_1\dot{\tilde{Q}}_F^\dagger; \qquad \dot{\tilde{Q}}_{2sed} = \dot{\tilde{Q}}_B + \frac{L_p^2}{L^2}\beta_2\dot{\tilde{Q}}_F \tag{52}$$

The corresponding action principle is

$$\frac{S}{\hbar} = \frac{1}{2}\int \frac{d\tau}{\tau_{Pl}}\; Tr\left[\frac{L_p^2}{L^2}\dot{\tilde{Q}}_{1sed}^\dagger\, \dot{\tilde{Q}}_{2sed}\right] \tag{53}$$

Each of the two dynamical variables is now a sedenion. We see here the power of the aikyon concept: it unifies bosons and fermions and has a very simple action principle which captures the standard model as well as gravity, and possibly also the four Higgs bosons. We will soon see that this Lagrangian actually already describes three fermion generations. For now however we will continue to work with the Lagrangian (45) as it makes it easier to make contact with known physics. This action is also reminiscent of string theory, where all elementary particles are different excitations of the string. But the similarity ends there: now the dynamics is trace dynamics, not quantum field theory (except in an emergent coarse-grained sense); the Fock space is constructed, not on the Minkowski vacuum, but on the algebraic vacuum defined in an octonion inspired Clifford algebra; the fundamental Hamiltonian is not self-adjoint; and the extra dimensions are not compactified.

From the work of Cartan (as quoted in [54], p. 923), it is known that the Lie algebra of G_2

possesses a symmetric invariant bilinear form

$$\beta \coloneqq x_0^2 + x_1 y_1 + x_2 y_2 + x_3 y_3 \tag{54}$$

This is the form left invariant by the lowest dimension representation of this Lie algebra, which happens to be a seven-dimensional complex representation. The scalar product β has real coefficients, which is consistent with the presence of the real coefficients α and L in our trace Lagrangian. Although we do not have a proof for this yet, the invariance of the trace Lagrangian under unitaries made from the generators of G_2 suggests that this trace is also Cartan's invariant bilinear form. As in known in trace dynamics, the existence of the Adler-Millard conserved charge is a consequence of this global unitary invariance, and this charge perhaps also has an intimate connection with the Lie algebra of G_2.

Unitary transformations of the dynamical variables are analogs of general coordinate transformations (in the Riemannian geometry of general relativity) in the present context of the non-commuting octonionic coordinates. The invariance of the trace Lagrangian, and the observation that this trace Lagrangian describes [quantum] gravity and the standard model (to be shown below) motivates us to make the following proposal. Physical laws are invariant under automorphisms ['general coordinate transformations'] of the octonionic coordinates, and this is responsible for the emergence of interactions as the (non-commutative) geometry of the octonionic space. We can elaborate on the apparent similarity of this assertion to the laws of special and general relativity, via the following observations. The extra four dimensions that are added on to space-time arise very naturally in our trace dynamics Lagrangian [22], and are analogous to the extra dimensions in Kaluza-Klein theories and give rise to the standard model forces. With the following difference from Kaluza-Klein theories: the entire 8-D space is now a non-commutative octonionic space, and the theory does not have to be quantised. Rather, because it is a trace dynamics, quantum (field) theory is emergent from our theory. The extra dimensions get suppressed in the classical limit [as we show below] because the fermions undergo spontaneous localisation under suitable conditions, and get localised to the real 4-D part of the octonionic space. Thus the symmetry group $SL(2,\mathbb{O})$ of the unified theory for one generation is obtained from complexified octonions, which unifies the Lorentz group with the standard model symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$. We will explain how gravity emerges only in the classical limit, as a consequence of spontaneous localisation of a large collection of aikyons. The octonionic space is the non-commutative analog of a space-time manifold. Every quantum elementary particle has its own such space - this describes

the small-scale structure of space-time. But one does not make a distinction between the particle and the octonionic space-time which it inhabits. Different aikyons interact via entanglement, a feature more fundamental than quantum theory. The latter inherits the property of entanglement from the underlying trace dynamics.

Automorphisms mix bosonic and fermionic terms in the trace Lagrangian. We recall though [22] that in our trace dynamics, bosonic matrices and fermionic matrices do not have a fixed spin (there is no Planck's constant in the underlying theory, it being only emergent, and the integral / half-integral spin of bosons / fermions is also emergent). So the mixing of the two kinds of terms is not a problem. However, assuming that the bilinear form (54) exists, it is always possible to write the Lagrangian in our chosen form, and then the following concrete interpretation of the various terms exists, leading to the association of this Lagrangian with the standard model.

We now work out the directions and components for each of the eight terms inside the trace Lagrangian (42). The first term T_1 proportional to $\dot{q}_B^\dagger \dot{q}_B$ comes from opening up $\dot{q}_B = \dot{q}_{Be0}\ e_0 + \dot{q}_{Be1}\ e_1 + \dot{q}_{Be2}\ e_2 + \dot{q}_{Be4}\ e_4$:

$$
\begin{aligned}
\dot{q}_B^\dagger \dot{q}_B &= \left[\dot{q}_{Be0}\ e_0 + \dot{q}_{Be1}^\dagger\ e_1 + \dot{q}_{Be2}^\dagger\ e_2 + \dot{q}_{Be4}^\dagger\ e_4\right] \times \left[\dot{q}_{Be0}\ e_0 + \dot{q}_{Be1}\ e_1 + \dot{q}_{Be2}\ e_2 + \dot{q}_{Be4}\ e_4\right] \\
&= \dot{q}_{Be0}^2 - \dot{q}_{Be1}^\dagger \dot{q}_{Be1} - \dot{q}_{Be2}^\dagger \dot{q}_{Be2} - \dot{q}_{Be4}^\dagger \dot{q}_{Be4} + \\
&\qquad \dot{q}_{Be0}\ [\dot{q}_{Be1}^\dagger\ e_1 + \dot{q}_{Be2}^\dagger\ e_2 + \dot{q}_{Be4}^\dagger\ e_4] + \ [\dot{q}_{Be1}\ e_1 + \dot{q}_{Be2}\ e_2 + \dot{q}_{Be4}\ e_4]\dot{q}_{Be_0}
\end{aligned}
\tag{55}
$$

Here the term $\dot{q}_{Be0}^2$ is assumed self-adjoint, being along the real direction, and is interpreted as the Higgs boson. The other cross-terms in the bi-product mutually cancel because of the product rule for octonions, and because in the trace Lagrangian, product of two bosonic Grassmann matrices commutes. The various terms in the resulting last line have the following interpretation, as we will show in detail below. It will be shown that $\dot{q}_B^\dagger \dot{q}_B$ describes Lorentz symmetry *and* the weak symmetry of the standard model! Indeed, these two symmetries arise together from the underlying theory, and are separated only when spontaneous localisation leads to the emergence of classical space-time. One might think of spontaneous localisation as a kind of spontaneous symmetry breaking which results from large-scale entanglement of fermions. The weak symmetry then becomes one of the three internal symmetries of the standard model. Thus the first four terms after the second equality above describe the Higgs boson and the three weak isospin bosons. The two terms in the last line describe two Lorentz bosons. In the classical limit, the trace of this term gives rise to the Einstein-Hilbert action and the short-range weak interaction, after symmetry breaking. The associated quaternionic spacetime is equivalent to 6D Minkowski spacetime, because

of the homomorphism $SL(2,\mathbb{H}) \sim SO(1,5)$.

We note that the matrix trace is to be taken as usual, keeping the octonion direction fixed. This will yield 'imaginary octonion direction based' terms in the trace Lagrangian as well as in the action, apart from the real terms. These imaginary terms go away in the emergent quantum theory and in the classical limit, leaving behind the desired Lagrangian. However, the imaginary terms are absolutely essential for describing unification in the underlying trace dynamics.

We now work out the components of the second term T_2 in the Lagrangian (42), which is proportional to $\dot{q}_B^\dagger \beta_2 \dot{q}_F$, using the coordinate assignment for the fermions shown in Eqn. (48). We have

$$
\begin{aligned}
\dot{q}_B \beta_2 \dot{q}_F = [\dot{q}_{Be0}\ e_0 + \dot{q}^\dagger_{Be1}\ e_1 + \dot{q}^\dagger_{Be2}\ e_2 + \dot{q}^\dagger_{Be4}\ e_4] \times \\
\left[\dot{q}_{Fe+}\ V_{e+} + \dot{q}_{Fu1}\ \frac{1}{4} V_{u1} + \dot{q}_{Fu2}\ V_{u2} + \dot{q}_{Fu3}\ V_{u3}\right] \\
= [\dot{q}_{Be0}\ e_0] \times \left[\dot{q}_{Fe+}\ V_{e+} + \dot{q}_{Fu1}\ V_{u1} + \dot{q}_{Fu2}\ V_{u2} + \dot{q}_{Fu3}\ V_{u3}\right] + \\
[\dot{q}^\dagger_{Be1}\ e_1] \times \left[\dot{q}_{Fe+}\ V_{e+} + \dot{q}_{Fu1}\ V_{u1} + \dot{q}_{Fu2}\ V_{u2} + \dot{q}_{Fu3}\ V_{u3}\right] + \\
[\dot{q}^\dagger_{Be2}\ e_2] \times \left[\dot{q}_{Fe+}\ V_{e+} + \dot{q}_{Fu1}\ V_{u1} + \dot{q}_{Fu2}\ V_{u2} + \dot{q}_{Fu3}\ V_{u3}\right] + \\
[\dot{q}^\dagger_{Be4}\ e_4] \times \left[\dot{q}_{Fe+}\ V_{e+} + \dot{q}_{Fu1}\ V_{u1} + \dot{q}_{Fu2}\ V_{u2} + \dot{q}_{Fu3}\ V_{u3}\right]
\end{aligned}
\tag{56}
$$

These sixteen terms have a simple interpretation. The first four represent what is usually referred to as Dirac-spin; the action of the conventional Dirac operator on four [positron, up quarks] of the eight fermions of a generation. (The other four fermions will come from the next term T_3 in the Lagrangian). The remaining 12 terms above describe the coupling of the three weak bosons to the four fermions. Remarkably, after spontaneous localisation leads to the emergence of classical space-time, the Dirac-spin terms separate from the rest, and give rise to the classical action for the relativistic point particle in general relativity. As a result, the weak interaction separates, and emerges as an internal symmetry, precisely because the directions (e_1, e_2, e_4) along which the weak bosons lie are imaginary octonionic directions.

In a way similar to the term T_2, we can now expand the third term T_3 of the Lagrangian, which is proportional to $\dot{q}_B \beta_1 \dot{q}_F^\dagger$. We have brought $\dot{q}_B$ to the front, which can be done inside the trace,

and without a change of sign, because $\dot{q}_B$ is bosonic. Thus we have,

$$
\begin{aligned}
\dot{q}_B\beta_1\dot{q}_F^\dagger = [\dot{q}_{Be0}\ e_0 + \dot{q}_{Be1}\ e_1 + \dot{q}_{Be2}\ e_2 + \dot{q}_{Be4}\ e_4]\times \\
\Big[q_{F\nu}\ V_\nu + q_{Fad1}\ V_{ad1} + q_{Fad2}\ V_{ad2} + q_{Fad3}\ V_{ad3}\Big] = \\
[\dot{q}_{Be0}\ e_0] \times \Big[q_{F\nu}\ V_\nu + q_{Fad1}\ V_{ad1} + q_{Fad2}\ V_{ad2} + q_{Fad3}\ V_{ad3}\Big] + \\
[\dot{q}_{Be1}\ e_1] \times \Big[q_{F\nu}\ V_\nu + q_{Fad1}\ V_{ad1} + q_{Fad2}\ V_{ad2} + q_{Fad3}\ V_{ad3}\Big] + \\
[\dot{q}_{Be2}\ e_2] \times \Big[q_{F\nu}\ V_\nu + q_{Fad1}\ V_{ad1} + q_{Fad2}\ V_{ad2} + q_{Fad3}\ V_{ad3}\Big] + \\
[\dot{q}_{Be4}\ e_4] \times \Big[q_{F\nu}\ V_\nu + q_{Fad1}\ V_{ad1} + q_{Fad2}\ V_{ad2} + q_{Fad3}\ V_{ad3}\Big]
\end{aligned}
\tag{57}
$$

In analogy with the term T_2, this term describes the action of the Dirac operator on the other four fermions of a generation [neutrino, anti-down quarks], and the action of the three weak bosons on these four fermions. Thus, together the terms T_1, T_2, T_3 of the Lagrangian describe the unified Lorentz-Weak symmetry. We will see below that this symmetry is described by the Clifford algebra $C\ell(6)$. This result is already there in the beautiful work of Stoica [45], and we will essentially repeat and report his, and Furey's, work below, in the context of our theory.

The other three terms of the Lagrangian, T_5, T_6, T_7 which we now analyse, analogously are associated with another copy of the Clifford algebra $C\ell(6)$. These describe the strong and electro-magnetic interactions of quarks and leptons, via $SU(3)_c \times U(1)_{em}$, as shown by Furey [25]. Why, one might ask, does this copy of $C\ell(6)$ behave differently from the first one? The answer lies in the fact that an octonion has only one real direction, not two. The first set of bosons associate with the directions (e_0, e_1, e_2, e_4) of which e_0 is real. The second set of bosons (to be analysed below) - the gluons and the photon - associate with the directions (e_3, e_5, e_6, e_7), all of which are imaginary. The presence of the real direction e_0 in the first set is what causes spontaneous localisation to occur, which results in the emergence of classical space-time and gravitation, and its separation from the three emergent internal symmetries.

Let us now look at term T_5 in (42), which is proportional to $q_B^\dagger q_B$. We recall that the assigned octonion directions for q_B are (e_3, e_5, e_6, e_7). Hence we get that

$$
\begin{aligned}
q_B^\dagger q_B &= \Big[q_{Be3}\ e_3 + q_{Be5}\ e_5 + q_{Be6}\ e_6 + q_{Be7}\ e_7\Big] \times \Big[q_{Be3}\ e_3 + q_{Be5}\ e_5 + q_{Be6}\ e_6 + q_{Be7}\ e_7\Big] \\
&= -q_{Be3}^\dagger q_{Be3} - q_{Be5}^\dagger q_{Be5} - q_{Be6}^\dagger q_{Be6} - q_{Be7}^\dagger q_{Be7}
\end{aligned}
\tag{58}
$$

All the cross terms cancel, because of the multiplication rule for the octonions. The four resulting terms, which describe the photon and gluons of three colours, should be compared and contrasted with the first set of bosonic terms derived in Eqn. (55). The difference is obvious: the first term there is positive, and there is also a cross-term, because of the presence of the real direction e_0. Therefore we now have a total of eight gluons $(q_B, q_B^\dagger)$ and they can be assigned the eight bosonic generators constructed by Furey for $SU(3)_c$ [25]. The $U(1)_{em}$ boson - the photon - is a number operator, and its contribution to the Lagrangian is the sum of the four terms after the second equality.

Next, let us look at the cross-terms $T_6 = q_B\beta_2 q_F$ and $T_7 = q_B^\dagger \beta_1 q_F^\dagger$. These describe the action of gluons and the photon on the quarks and leptons:

$$\begin{aligned} q_B\beta_2 q_F = [q_{Be3}\ e_3 + q_{Be5}\ e_5 + q_{Be6}\ e_6 + q_{Be7}\ e_7]\times \\ \left[q_{Fe-}\ V_{e-}^* + q_{Fau1}\ V_{au1}^* + q_{Fu2}\ V_{au2}^* + q_{Fu3}\ V_{au3}^*\right] \\ = [q_{Be3}\ e_3]\times\left[q_{Fe-}\ V_{e-}^* + q_{Fau1}\ V_{au1}^* + q_{Fau2}\ V_{au2}^* + q_{Fau3}\ V_{au3}^*\right]+ \\ [q_{Be5}\ e_5]\times\left[q_{Fe-}\ V_{e-}^* + q_{Fau1}\ V_{au1}^* + q_{Fau2}\ V_{au2}^* + q_{Fau3}\ V_{au3}^*\right]+ \\ [q_{Be6}\ e_6]\times\left[q_{Fe-}\ V_{e-}^* + q_{Fau1}\ V_{au1}^* + q_{Fau2}\ V_{au2}^* + q_{Fau3}\ V_{au3}^* + \right]+ \\ [q_{Be7}\ e_7]\times\left[q_{Fe-}\ V_{e-}^* + q_{Fau1}\ V_{au1}^* + q_{Fau2}\ V_{au2}^* + q_{Fau3}\ V_{au3}^*\right] \end{aligned} \tag{59}$$

Out of the four components $(q_{Be3}\ e_3 + q_{Be5}\ e_5 + q_{Be6}\ e_6 + q_{Be7}\ e_7)$, the one representing the photon will be moved, along with terms representing its action on the fermions, to join the weak interaction terms, after spontaneous localisation separates the Dirac operator terms from the weak interaction terms. Thus, $U(1)_{em}$ joins with $SU(2)_W$ to form $SU(2)_L \times U(1)_Y$.

$$\begin{aligned} q_B^\dagger\beta_1 q_F^\dagger = [q_{Be3}\ e_3 + q_{Be5}\ e_5 + q_{Be6}\ e_6 + q_{Be7}\ e_7]\times \\ \left[q_{Fa\nu}\ V_{a\nu}^* + q_{Fd1}\ V_{d1}^* + q_{Fd2}\ V_{d2}^* + q_{Fd3}\ V_{d3}^*\right] = \\ [q_{Be3}^\dagger\ e_3]\times\left[q_{Fa\nu}\ V_{a\nu}^* + q_{Fd1}\ V_{d1}^* + q_{Fd2}\ V_{d2}^* + q_{Fd3}\ V_{d3}^*\right]+ \\ [q_{Be5}^\dagger\ e_5]\times\left[q_{Fa\nu}\ V_{a\nu}^* + q_{Fd1}\ V_{d1}^* + q_{Fd2}\ V_{d2}^* + q_{Fd3}\ V_{d3}^*\right]+ \\ [q_{Be6}^\dagger e_6]\times\left[q_{Fa\nu}\ V_{a\nu}^* + q_{Fd1}\ V_{d1}^* + q_{Fd2}\ V_{d2}^* + q_{Fd3}\ V_{d3}^*\right]+ \\ [q_{Be7}^\dagger\ e_7]\times\left[q_{Fa\nu}\ V_{a\nu}^* + q_{Fd1}\ V_{d1}^* + q_{Fd2}\ V_{d2}^* + q_{Fd3}\ V_{d3}^*\right] \end{aligned} \tag{60}$$

Lastly, we look at the terms T_4 and T_8 in the Lagrangian, which can be together written as

$$\begin{aligned}
&T_4 + T_8 = \frac{L_P^4}{L^4}\left[\beta_1 \dot{q}_F^\dagger \beta_2 \dot{q}_F - \frac{\alpha^2}{L^2}\beta_1 q_F^\dagger \beta_2 q_F\right] = \\
&\frac{1}{8}\frac{L_P^4}{L^4}V_{e+}\left[\left(\dot{q}_\nu \dot{q}_{Fe+} + \dot{q}_{Fad1}\dot{q}_{Fu1} + \dot{q}_{Fad2}\dot{q}_{Fu2} + \dot{q}_{Fad3}\dot{q}_{Fu3}\right) - \right. \\
&\left.\frac{\alpha^2}{L^2}\left(q_{Fa\nu}q_{Fe-} + q_{Fd1}q_{Fau1} + q_{Fd2}q_{Fau2} + q_{Fd3}q_{Fau3}\right)\right]
\end{aligned} \tag{61}$$

Here we see the action of fermions onto themselves. The bi-octonion structure is missing here, because we are not retaining total time derivatives in the Lagrangian. However the bi-octonionic structure is evident in the form (50) of the Lagrangian.

Having introduced the various terms in the Lagrangian, we now justify their claimed relation with the standard model. For the most part, the work on division algebras and Clifford algebras by Furey [25], Stoica [45] and several other researchers, essentially accomplishes this already. We report their work below, to justify the connection of our Lagrangian with division algebras and the standard model. The new part is the direct evidence for the presence of the gravito-weak symmetry in our Lagrangian, whose existence has been conjectured by several researchers, notably by Onofrio [55, 56], [57, 58], and also hinted at by Stoica [45].

VIII. DIVISION ALGEBRAS, AND ONE GENERATION OF STANDARD MODEL FERMIONS

1. *The Lorentz symmetry and the Clifford algebra $C\ell(2)$*

If we consider the four quaternion components of $\dot{q}_B$, whose bi-product lies in the Top-Left part of the multiplication table in Fig. 1, we recall that the term $T_1 \propto \dot{q}_B^\dagger \dot{q}_B$ is given by Eqn. (55). For the special case that each of the four bosonic components is equal to the identity matrix, the sum is of the form $I \times (1, -1, -1, -1)$. This form is Lorentz invariant and the Lorentz algebra $SL(2, C)$ is known to be generated by complex quaternions. Because it aids understanding the Clifford algebra $C\ell(6)$ we recall that complex quaternions give a faithful representation of $C\ell(2)$. This can be arrived at using the following two fermionic ladder operators:

$$\alpha_0 = \frac{1}{2}(ie_1 - e_2); \qquad \alpha_0^\dagger = \frac{1}{2}(ie_1 + e_2); \qquad \alpha_0^2 = \alpha_0^{\dagger 2} = 0, \{\alpha_0, \alpha_0^\dagger\} = 1 \tag{62}$$

Spinors can be constructed as minimal left ideals of $C\ell(2)$ by using the idempotent $V \equiv \alpha\alpha^\dagger$ which acts like the vacuum state. Left multiplying V by the Clifford algebra of complex quaternions is the minimal left ideal, which happens to be a 2-D complex space spanned by V and $\alpha^\dagger V$. Their linear combination gives left-handed Weyl spinors under the Lorentz algebra $SL(2, C)$. Similarly, $C\ell(2)$ acting on the complex-conjugate V^* gives rise to the space $(V^*, \alpha V^*)$ whose linear combination gives rise to right-handed Weyl spinors [25]. The left-handed Weyl spinors and the right-handed Weyl spinors are simply complex conjugates of each other.

This construction, though elementary, has deep significance and implications for our theory, considering that Furey [25] then develops a completely analogous construction for the Clifford algebra $C\ell(6)$, from the octonion algebra, to describe quarks and leptons and their unbroken $SU(3)_c \times U(1)_{em}$ symmetry. This puts the Weyl spinors on the same footing as the fermions, as if the former were particles too. And that is how it turns out to be, as we will soon see. This also reinforces the aikyon concept, which puts space-time and matter on the same footing, at the Planck scale. This interpretation is further strengthened because Stoica [45] incorporates the Lorentz algebra and the weak symmetry together, to form the other copy of $C\ell(6)$, when describing the symmetries of quarks and leptons in the standard model. This leads us to propose the gravito-weak interaction.

2. *The unbroken electro-colour symmetry and the Clifford algebra $C\ell(6)$*

This section follows [25], and is a (condensed) repetition of the results therein, to put those results in the context of our theory, and to link up to the next sub-section. The reader is referred to [25] for further details on arriving at electro-colour symmetry, from the algebra of octonions.

The standard model symmetries are being arrived at by multiplying the algebra onto itself. But why does this scheme work? The answer becomes obvious when we take note that in our Lagrangian, every term is quadratic or bilinear. Therefore when we compute the terms of the Lagrangian, written out in octonionic space, the algebra inevitably acts on itself, and that dictates the symmetries. The Grassmann matrices simply go for a ride when the symmetries of the Lagrangian are being determined, and the role of the matrices comes to the fore in the dynamics and in the emergent theory. The algebra has no further role to play in the emergent quantum theory, nor in the classical limit. But by then the algebra has already made its mark, and the correctly predicted properties of the standard model are evidence that division algebras are at work at the Planck scale.

Multiplication of octonions is not associative, whereas Clifford algebras are associative. Hence the former cannot give a faithful representation of the latter. Nonetheless, there is a trick using which $C\ell(6)$ can be built from the complex octonions. The idea is to think of a chain of left multiplications in the (non-associative) algebra as a way to generate maps from one element in the algebra to another. And maps are necessarily associative. For the complex quaternions, this yields nothing new: an associative algebra leads to maps which are necessarily associative. However, for complex octonions this works wonders, thanks to their amazing mathematical properties. It turns out that to describe the most general chain of left multiplying elements of the algebra [and these chains then serve as maps which are associative], one does not need chains of length more than three, and these chains, acting as maps, behave as elements of a Clifford algebra. It is easily shown that there are sixty-four such independent complex-valued octonionic chains [one of length zero, seven of length one, twenty-one of length two, and thirty-five of length three]. They can be thought of as being equivalent to 8x8 complex matrices, and give a faithful representation of $Cl(6)$.

Then, just like in the case of $C\ell(2)$ above, where the complex quaternions were used to construct a new basis, a new basis is constructed now, from one-vectors, noting that unity is the zero vector, and product of any six imaginary directions, say $(e_1, e_2, e_3, e_4, e_5, e_6)$ gives the seventh one, in this case e_7. So the six imaginary directions are used to make six fermionic ladder operators, and these are the following (it being understood that these form octonionic chains/maps, always acting on an octonion, though not shown here explicitly)

$$\begin{aligned} \alpha_1 = \frac{1}{2}[-e_5 + ie_4]; \qquad \alpha_2 = \frac{1}{2}[-e_3 + ie_1]; \qquad \alpha_3 = \frac{1}{2}[-e_6 + ie_2]; \\ \alpha_1^\dagger = \frac{1}{2}[e_5 + ie_4]; \qquad \alpha_2^\dagger = \frac{1}{2}[e_3 + ie_1]; \qquad \alpha_3^\dagger = \frac{1}{2}[e_6 + ie_2] \end{aligned} \tag{63}$$

it being understood that all multiplications are left multiplications. These ladder operators satisfy the Clifford algebra

$$\{\alpha_i, \alpha_j\} = 0; \qquad \{\alpha_i^\dagger, \alpha_j^\dagger\} = 0; \qquad \{\alpha_i, \alpha_j^\dagger\} = \delta_{ij} \tag{64}$$

The number operator is defined as usual

$$N = \sum_1^3 \alpha_i^\dagger \, \alpha_i \tag{65}$$

The ladder operators have a unitary symmetry $U(3)$, which rotates the lowering operators amongst themselves, and the raising operator amongst themselves. And we know that $U(3) = SU(3) \times$

$U(1)/\mathbb{Z}_3$. It so happens that $SU(3)$ is a sub-group of G_2 which holds one of the imaginary units constant. Whereas $U(1)$ is generated by the number operator N above.

As before, one now constructs minimum left ideals of the Clifford algebra $C\ell(6)$, by first defining the idempotent V (the projector), which acts as the 'vacuum', as follows:

$$V = \alpha_1\alpha_2\alpha_3\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger = \frac{1}{2}(1 + ie_7) \qquad \text{[Neutrino]} \tag{66}$$

The particular form $(1 + ie_7)/2$ is of course specific to the coordinates we have chosen (though generic enough, amounting to setting $f = 1$ in Furey's notation, and not restricting any of the conclusions) and we shortly explain why this is the neutrino. The minimal left ideal is obtained by left multiplying on the vacuum V by the Clifford algebra; this yields an eight complex dimensional space denoted S^u and spanned by the following basis vectors:

$$\begin{aligned}
V &= \frac{1}{2}(1 + ie_7) \qquad [V_\nu \text{ Neutrino}] \\
\alpha_1^\dagger V = \frac{1}{2}(e_5 + ie_4) \times V &= \frac{1}{2}(e_5 + ie_4) \qquad [\mathrm{V_{ad1}} \text{ Anti} - \text{down quark}] \\
\alpha_2^\dagger V = \frac{1}{2}(e_3 + ie_1) \times V &= \frac{1}{2}(e_3 + ie_1) \qquad [\mathrm{V_{ad2}} \text{ Anti} - \text{down quark}] \\
\alpha_3^\dagger V = \frac{1}{2}(e_6 + ie_2) \times V &= \frac{1}{2}(e_6 + ie_2) \qquad [\mathrm{V_{ad3}} \text{ Anti} - \text{down quark}] \\
\alpha_3^\dagger\alpha_2^\dagger V &= \frac{1}{2}(e_4 + ie_5) \qquad [\mathrm{V_{u1}} \text{ Up quark}] \\
\alpha_1^\dagger\alpha_3^\dagger V &== \frac{1}{2}(e_1 + ie_3) \qquad [\mathrm{V_{u2}} \text{ Up quark}] \\
\alpha_2^\dagger\alpha_1^\dagger V &= \frac{1}{2}(e_2 + ie_6) \qquad [\mathrm{V_{u3}} \text{ Up quark}] \\
\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger V &= -\frac{1}{2}(i + e_7) \qquad [\mathrm{V_{e+}} \text{ Positron}]
\end{aligned} \tag{67}$$

Similarly, one can define the space S_d by acting the algebra on the conjugate idempotent V^*,

spanned by the following eight vectors:

$$
\begin{aligned}
V^* &= \frac{1}{2}(1 - ie_7) \qquad [V_{a\nu} \text{ Anti} - \text{neutrino}]\\
\alpha_1 V^* = \frac{1}{2}(-e_5 + ie_4) \times V^* &= \frac{1}{2}(e_5 - ie_4) \qquad [V^*_{d1} \text{ Down quark}]\\
\alpha_2 V^* = \frac{1}{2}(-e_3 + ie_1) \times V^* &= \frac{1}{2}(e_3 - ie_1) \qquad [V^*_{d2} \text{ Down quark}]\\
\alpha_3 V^* = \frac{1}{2}(-e_6 + ie_2) \times V^* &= \frac{1}{2}(e_6 - ie_2) \qquad [V^*_{d3} \text{ Down quark}]\\
\alpha_2\alpha_3 V^* &= \frac{1}{2}(e_4 - ie_5) \qquad [V^*_{au1} \text{ Anti} - \text{up quark}]\\
\alpha_3\alpha_1 V^* &== \frac{1}{2}(e_1 - ie_3) \qquad [V^*_{au2} \text{ Anti} - \text{up quark}]\\
\alpha_1\alpha_2 V^* &= \frac{1}{2}(e_2 - ie_6) \qquad [V^*_{au3} \text{ Anti} - \text{up quark}]\\
\alpha_1\alpha_2\alpha_3 V^* &= -\frac{1}{2}(-i + e_7) \qquad [V^*_{e-} \text{ Electron}]
\end{aligned}
\tag{68}
$$

The action of the algebra shows that anti-particles are simply complex conjugates of the particles. Now one needs to justify the particle labels in the above basis. The ladder operators transform under $U(3)$ symmetry [made of $SU(3)$ and $U(1)$)], and hence so do the vectors in the above basis. Consider the action of $SU(3)$ first. Now we have noted that $SU(3)$ is a sub-group of G_2 obtained by keeping one of the imaginary directions, say e_7, fixed. The fourteen generators of G_2 can be expressed as chains of octonion elements acting on the octonions. One could ask how out of all of G_2, the groups $SU(3)$ and $U(1)$ are selected. It has been shown that the generating space of a Clifford algebra $C\ell(n)$ with n even can always be partitioned into two maximal totally isotropic subspaces (MTIS) [25] and in the case of $C\ell(6)$ these are: one MTIS spans $(\alpha_1, \alpha_2, \alpha_3)$ and the other spans $(\alpha_1^\dagger, \alpha_2^\dagger, \alpha_3^\dagger)$. If this separation is to be preserved under transformations of the ladder operators, the MTIS are precisely those generated by the Lie algebra of $SU(3)$ and $U(1)$. The $SU(3)$ generators made from these ladder operators are given in Eqn. (6.26) of [25] and we do not repeat them here. The generator for $U(1)$ is given by $Q = N/3$ where N is the number operator given above.

Now, under $SU(3)$, the state V in S^u translates as a singlet, the next three out of those translate as anti-triplets, the three after those as triplets, and the last one as a singlet. Analogously, in S^d, the first state V^* transforms as a singlet, the next three as a triplet, the three after those as anti-triplets, and the last one as a singlet. Next, one finds the eigenvalues of the operator Q and finds them to be $(0, 1/3, 1/3, 1/3, 2/3, 2/3, 2/3, 1)$ and the eigenvalues of the eight states listed under S^u are precisely these values, in this very order. Analogously, in S^d, the charges for the eight states

come out to be $(0, -1/3, -1/3, -1/3, -2/3, -2/3, -2/3, -1)$. We can now identify these objects given these properties. In S^u, a singlet under $SU(3)$ with zero charge is either a neutrino or an anti-neutrino: it turns out to be the neutrino as we justify in the next sub-section. The three anti-triplets with charge $1/3$ are anti-down quarks, the three triplets with charge $2/3$ are quarks, and the singlet with charge 1 is the positron. Similarly, in S^d, one identifies an anti-neutrino, three down quarks, three anti-up quarks, and the electron.

It is a great triumph of Furey's work that she is able to derive the quantisation of electric charge, and three colours, for quarks and leptons, simply from the algebra of complex octonions acting on themselves. It is a strong hint that fundamentally the fermions and exchange bosons of the standard model live in an octonionic space, and their dynamics is described by a Lagrangian with a quadratic form. The existence of such a dynamics is borne out by the Lagrangian we have constructed. Our Lagrangian was not constructed to explain the significance of division algebras. Rather, our goal stemmed from quantum foundations: to achieve a reformulation of quantum field theory which does not depend on classical time. This led us to the theory of trace dynamics, and also to Connes' non-commutative geometry (this latter for including gravity into trace dynamics, as a matrix dynamics). This is how we constructed an appropriate trace dynamics Lagrangian for gravity coupled to fermions, ensuring that quantum theory is emergent, and that it has the desired classical limit [7]. We then showed how Yang-Mills fields can be brought into the Lagrangian, in the conventional spirit of modifying the Dirac operator [21]. Only subsequently it was realised that a fundamental explanation of spin [59] after including Yang-Mills interactions strongly indicates the doubling of space-time dimensions from four to eight. This is how octonions were implicated in our theory. The successful merging of two apparently disparate investigations [division algebras and standard model, versus trace dynamics with gravity] strongly suggests that both the investigations are on the right track. The two investigations complement each other.

This is how the terms T_4, T_5, T_6, T_7 and T_8 in our Lagrangian relate to the algebra of complex octonions, and to the electro-color symmetry of the standard model. We now show that another copy of $C\ell(6)$ relates the terms (T_1, T_2, T_3) in the Lagrangian to the other two interactions, gravitation and the weak force, via the Lorentz-weak symmetry, which we newly propose in our approaach. It turns out that these three terms will force us to extend the algebra to sedenions, that being the only way to include gravity in the standard model. But as a bonus, we will get three fermion generations. The existence of three generations is an inevitable consequence of unifying gravity with the standard model.

We start by summarising our new findings and then we justify them in detail [50]. We have seen above that the symmetry group within G_2, which describes the electro-color symmetry and the fermion properties, is $SU(3) \times U(1)$. It is the element-wise stabiliser group of octonions. It turns out that the symmetry group which describes the first three terms (T_1, T_2, T_3) of our Lagrangian is (obtained from the complexification of) $\mathrm{Stab}_{G_2}(\mathbb{H})$, the stabiliser group of the quaternions inside the octonions. This group happens to be $SO(4)$. It is the other maximal sub-group of G_2, beisdes $SU(3)$. This is the group which describes the gravito-weak symmetry. It is the group generated by the Clifford algebra $C\ell(6)$ which Stoica [45] constructs to describe the weak and Lorentz sector. A sub-group of the stabiliser group $\mathrm{Stab}_{G_2}(\mathbb{H})$ (this being $SO(4)$), is the element-wise stabiliser group of quaternions, $\mathrm{Fix}_{\mathrm{G}_2}(\mathbb{H})$, which happens to be $SU(2)$. This $SU(2)$ describes the weak symmetry, corresponding to a $C\ell(4)$ Clifford algebra. Thus, just as the element-wise stabiliser group $SU(3)$ of octonions describes the electro-colour symmetry, the element-wise stabiliser group $SU(2)$ of quaternions describes the weak symmetry. The groups $SU(3)$ and $SO(4)$ thus constructed have an intersection which is a $U(2)$ group. The $SU(2)$ of weak symmetry is the simple part of $U(2)$ and is also a normal sub-group of $SO(4)$. In this manner, the weak symmetry is a part of both the standard model symmetries and the space-time gravito-weak symmetry. The $SO(4)$ is a group extension of $SO(3)$ - the automorphism group of quaternions (i.e. $SO(3)$) - by the $SU(2)$. We have seen that complex quaternions generate Lorentz symmetry. Thus, $SU(2)$ works as a bridge to unify the standard model with the Lorentz symmetry and thereby with gravity. Also, the gravito-weak symmetry correctly maps the projector of S^u basis (see previous section) to the particles in S^d. It explains why weak interactions violate parity. The two Clifford algebras $C\ell(6)$ constructed by Stoica are not completely different - they happen to have the $SU(2)$ weak symmetry overlap, in the group theoretic sense just mentioned. Thus effectively, the Clifford algebra they amount to is $C\ell(8)$. This then is the Clifford algebra which describes the unification of the standard model with Lorentz symmetry. However, $C\ell(8)$ cannot be constructed from the algebra of octonions. Therefore, following Gillard and Gresnigt [46], we are compelled to go beyond division algebras, onto complex sedenions. Because the minimal left ideals that the complex sedenions generate leads to a useful $C\ell(14)$. Moreover, the automorphism group of sedenions is essentially three copies of the automorphism group G_2 of octonions, suggesting a way to get the three fermion generations [46]. Remarkably, these three copies of G_2 have an intersection, which happens precisely to be the stabiliser group of the quaternions! Thus the gravito-weak symmetry is shared amongst the three

generations, only the electro-colour part differs. The theory has four extra terms yet unaccounted for, which could describe the Higgs bosons. We predict two new spin one bosons, the Lorentz bosons [so named recently by Cahill [60]], which describe the quantisation of the Lorentz symmetry. This, and not the graviton, is the gravitational analog of the photon. After the universe undergoes spontaneous localisation and classical space-time emerges, the electro-weak symmetry becomes part of the internal symmetries of the standard model. Gravitation emerges essentially as a gauging of the Lorentz symmetry possessed by the individual aikyons. We note that above we have described only one aikyon, and the various bosons and leptons are its different manifestations.

We now explain in detail as to how we came to these conclusions. The group theoretic properties of G_2 mentioned above can be found, amongst other places, in the nice summary available at this link, along with references. The cartoon below attempts to describe the relative role of the various symmetry groups. We rely heavily on the three recent and important papers by Furey [26], by Gillard and Gresnigt [46], and by Stoica [45]. Incidentally, all these three papers came out as recently as 2019, and without these papers the present work would be impossible.

Stoica constructs two copies of $C\ell(6)$ [no division algebras involved; only Clifford algebras] one of which describes the electro-colour symmetry, and matches precisely with the $C\ell(6)$ which Furey constructs and which we described above. The more interesting part is the second copy of $C\ell(6)$ constructed by Stoica. Four of the six generators for the Clifford algebra properly describe the action of $SU(2)$ weak symmetry on the quarks and leptons. The strange part is this. While these generators are not linear combinations of the generators of the other $C\ell(6)$, and in that sense not dependent on them, nonetheless they bear an intricate mathematical relation to the electro-color generators. This is extremely surprising and suggestive. It is telling that the electro-colour part somehow knows about the weak symmetry [as Furey notes too, emphatically]. Moreover, while Stoica correctly adds the Lorentz symmetry to the $SU(2)$ to make the second copy of $C\ell(6)$, it is discomforting that Lorentz symmetry has to be added on by hand. This is a strong indication that the weak and Lorentz symmetries must be unified in a fundamental way, using division algebras. And above all, the terms T_1, T_2, T_3 in the Lagrangian are pointing towards Lorentz-weak unification - this is evident from their composition of these terms, their corresponding counter-part having been taken up by electro-colour.

Furey also constructs a $C\ell(4)$, from the generators of the electro-colour $C\ell(6)$, and these correctly describe the right action of $SU(2)_{weak}$ on quarks and leptons. Let us recall how this was done. Given the nilpotent $\omega = \alpha_1\alpha_2\alpha_3$ made from the electro-colour generators of the previous

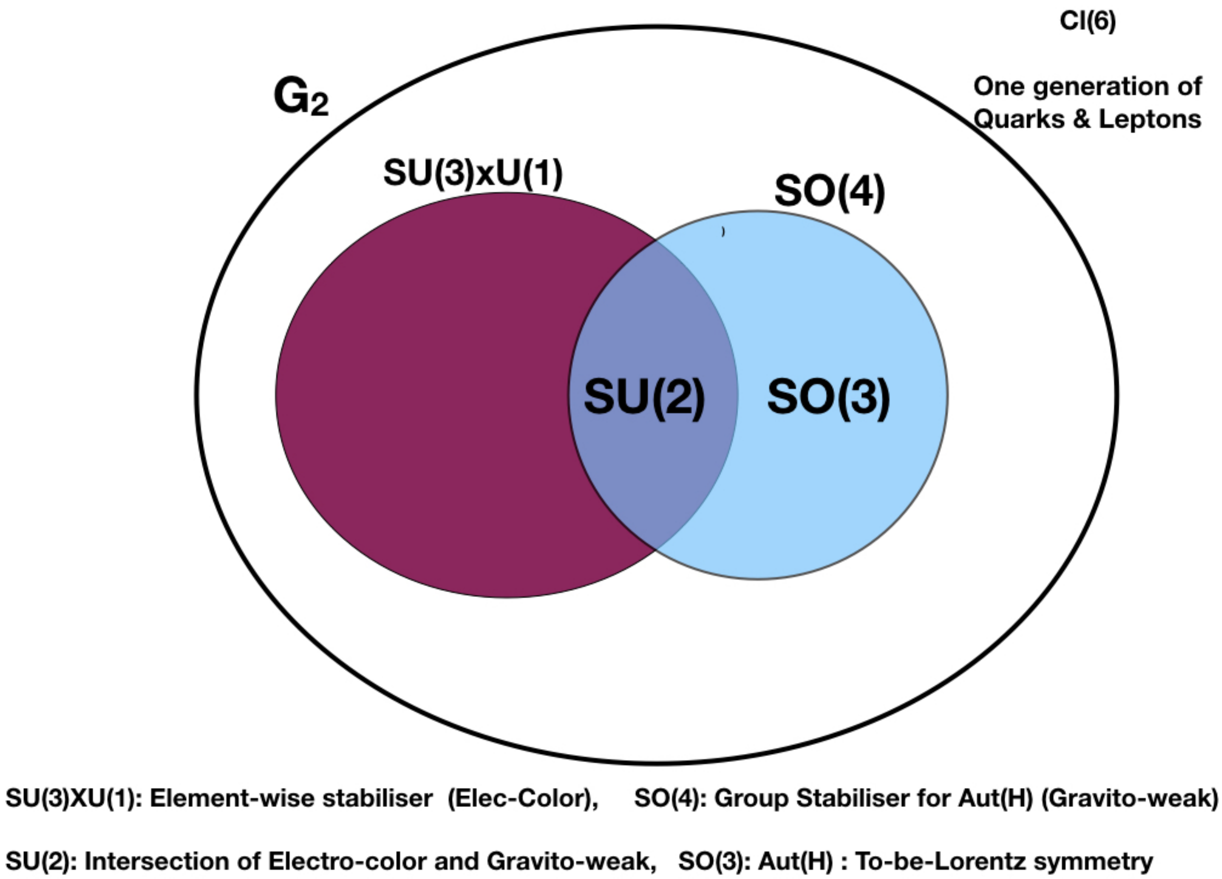


FIG. 2.

The two maximal sub-groups of G_2 whose intersection is $U(2) \sim SU(2) \times U(1)$. This diagram shows the unification of the standard model with Lorentz symmetry and hence with gravity. The element stabiliser group of G_2 generates the electro-color symmetry $SU(3)$. The group stabilizer of Aut($\mathbb{H}$) generates the gravito-weak symmetry $SO(4) \sim SU(2) \times SU(2)$. Their intersection is the weak symmetry $SU(2)$ enhanced by $U(1)_{em}$. Gravito-weak extends $SO(3)$ by the weak symmetry because $SO(4)$ is a group extension of $SO(3)$ by $SU(2)$. Lorentz symmetry $SL(2,C)$ is constructed from generators of $C\ell(2)$ made from $\mathbb{C} \times \mathbb{H}$ [complexification of $SO(3)$, i.e. of Aut($\mathbb{H}$).] [50]

section, the following $C\ell(4)$ generators are constructed from them:

$$(\tau_1 i\epsilon_1, \tau_2 i\epsilon_1, \tau_3 i\epsilon_1, i\epsilon_2),$$

where $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$ are the basis vectors of a quaternion, and where $\tau_1 \equiv \omega + \omega^\dagger, \tau_2 \equiv i\omega - i\omega^\dagger, \tau_3 \equiv \omega\omega^\dagger - \omega^\dagger\omega$. These generators are then written in a new basis $[\beta_1, \beta_2, \beta_1^\ddagger, \beta_2^\ddagger]$ and are fermionic ladder

operators, where

$$\beta_1 \equiv (i\epsilon_2 + i\epsilon_1\tau_3); \qquad \beta_2 \equiv \omega^\dagger i\epsilon_1 \tag{69}$$

[These ladder operators are analogous to the $(\omega_u, \omega_u^\dagger, \omega_d, \omega_d^\dagger)$ constructed by Stoica [45], but not the same ones]. Here, ‡ stands for simultaneous complex conjugation, quaternion conjugation and octonion conjugation. These generators can be shown to form the Clifford algebra $C\ell(4)$. An idempotent is constructed, and the ladder operators and minimal right ideals transform the leptons as expected for weak interactions, under an $SU(2)$ symmetry constructed from the following three $SU(2)$ generators:

$$T_1 \equiv \tau_1(1 + i\epsilon_3); \qquad T_2 \equiv \tau_2(1 + i\epsilon_3); \qquad T_3\tau_3 \equiv \tau_3(1 + i\epsilon_3) \tag{70}$$

where ϵ_3 is the third quaternion component, which was not used in making the β generators. The chirality property of the leptons is automatically recovered under the application of this $SU(2)$.

Let us look at this remarkable construction more closely, as it will guide us to the stabiliser group $SO(4) \sim SU(2) \times SU(2)$ of quaternions and the proposed gravito-weak symmetry. The β generators are made from G_2 automorphisms acting on the α generators of the electro-colour symmetry, and from two of the three vectors of a quaternion. Let us *assume* that this quaternion belongs to the quaternion sub-algebra of the very octonions we are studying and which describe the standard model (i.e. $\epsilon \in e$). What are the implications of this assumption? These β generators result from automorphisms which belong to the $SU(3)$ group shown above in Fig. 2. Furthermore thery are made from acting on *only two* of the three quaternionic elements of the said quaternion. For the moment, just to indicate where we are headed, let us assume that τ_1, τ_2 and τ_3 are different from the quaternionic components ϵ_1 and ϵ_2. Because ϵ_3 is not transformed by the β automorphisms, these automorphisms belong to the element-preserving group $SU(2)$ of the quaternions. [Just as $SU(3)$ is the element-preserving group of the octonions]. This appears to be the reason why the above construction works. So we can now look beyond this specific construction, and propose the following. Consider the action on quaternions, of those automorphisms inside the chosen $SU(3)$, which belong to the element-preserving group $SU(2)$ of quaternions. Ladder operators made from these automorphisms will transform under the (τ_1, τ_2, τ_3) (which are used to construct the $SU(2)$ generators), precisely as under an $SU(2)$ symmetry. We propose to identify this with the weak symmetry of the standard model. Weak interactions are that part of the electro-

colour symmetry which belong to the element-preserving group $SU(2)$ of the quaternions inside the element-preserving group $SU(3)$ of the octonions. The physical reason behind this mathematical proposal remains to be understood, just as we do not know why electro-colour symmetry is described by the element preserving group of the octonions. The explanation will possibly come from the unified theory, which we will describe below, shortly.

Our claim finds strong support in the recent research of other workers, on the maximal subgroups of compact exceptional Lie-groups, and their relevance for the standard model. These developments are based on the Borel - di Siebenthal [61] theory for classification of such groups. See especially the works of Todorov and Drenska [33], Todorov and Dubois-Violette [52], and Baez and Huerta [41]. Section 2.1 of the first of these papers has an elegant proof that the element-preserving sub-group inside G_2 is $SU(3)$ and the stabiliser group of the quaternions inside the octonions is $SU(2) \times SU(2)/\mathbb{Z}_2$. Eqn. (4.2) of the paper by Todorov and Dubois-Violette [52] notes that the intersection of these two groups is $U(2)$, which happens to be the gauge-group for the Weinberg-Salam model, and has the $SU(2)_L$ sub-group inside it. Thus, following our proposal in the previous paragraph, we re-state that this $U(2)$ should indeed be seen as a part of the electro-colour symmetry, somehow suggesting that electro-weak follows from electro-colour. We will have more to say about this when we return to the terms T_1, T_2, T_3 in our Lagrangian.

We now propose the gravito-weak symmetry as the (complexification of) the automorphism-group of the other maximal-subgroup of G_2, namely $SU(2)\times SU(2)/\mathbb{Z}_2$. This group is also the group extension, by the afore-mentioned $SU(2)$, of the automorphism group $\mathrm{Aut}(\mathbb{H})$ of the quaternions. It is also the stabiliser group of quaternions, which means that automorphisms of G_2 belonging to this group, when acting on quaternions, will send them to other quaternions. This safety-net for the quaternions within the octonions is absolutely essential for the emergence of classical space-time [whose Lorentz symmetry group is $\mathbb{C} \times \mathbb{H}$; automorphisms of this group lead to general relativity, as we see below.]

We recall our Lagrangian (45) below, now without the adjointness condition imposed on q_B. Only $\dot{q}_B$ will be assumed to be self-adjoint.

$$\frac{S}{\hbar} = \int \frac{d\tau}{t_{Pl}}\, Tr \Bigg\{ \frac{L_P^2}{L^2} \left[\left(\dot{q}_B^\dagger + i\frac{\alpha}{L} q_B^\dagger \right) + \frac{L_P^2}{L^2}\beta_1 \left(\dot{q}_F^\dagger + i\frac{\alpha}{L} q_F^\dagger \right) \right] \times \left[\left(\dot{q}_B + i\frac{\alpha}{L} q_B^\dagger \right) + \frac{L_P^2}{L^2}\beta_2 \left(\dot{q}_F + i\frac{\alpha}{L} q_F \right) \right] \Bigg\} \tag{71}$$

which can also be written as

$$\frac{S}{\hbar} = \int \frac{d\tau}{\tau_{Pl}}\, Tr\bigg\{ \frac{L_P^2}{L^2} \bigg[\bigg(\dot{q}_B^\dagger + \frac{L_P^2}{L^2}\beta_1 \dot{q}_F^\dagger \bigg) + i\frac{\alpha}{L} \bigg(q_B^\dagger + \frac{L_P^2}{L^2}\beta_1 q_F^\dagger \bigg) \bigg] \times \bigg[\bigg(\dot{q}_B + \frac{L_P^2}{L^2}\beta_2 \dot{q}_F \bigg) + i\frac{\alpha}{L} \bigg(q_B + \frac{L_P^2}{L^2}\beta_2 q_F \bigg) \bigg] \bigg\} \tag{72}$$

The sum of the first three terms is assumed to be invariant under automorphisms belonging to the stabiliser group of quaternions $\mathrm{Stab}_{G2}(\mathbb{H})$. This is the gravito-weak symmetry. We recall that the bosonic indices are the quaternionic part of octonion indices and run from zero to three. The bosonic kinetic energy term (55) is $\dot{q}_B = \dot{q}_{Be0}\, e_0 + \dot{q}_{Be1}\, e_1 + \dot{q}_{Be2}\, e_2 + \dot{q}_{Be4}\, e_4$:

$$\begin{aligned} \dot{q}_B^\dagger \dot{q}_B &= \Big[\dot{q}_{Be0}\, e_0 + \dot{q}_{Be1}^\dagger\, e_1 + \dot{q}_{Be2}^\dagger\, e_2 + \dot{q}_{Be4}^\dagger\, e_4 \Big] \times \Big[\dot{q}_{Be0}\, e_0 + \dot{q}_{Be1}\, e_1 + \dot{q}_{Be2}\, e_2 + \dot{q}_{Be4}\, e_4 \Big] \\ &= \dot{q}_{Be0}^2 - \dot{q}_{Be1}^\dagger \dot{q}_{Be1} - \dot{q}_{Be2}^\dagger \dot{q}_{Be2} - \dot{q}_{Be4}^\dagger \dot{q}_{Be4} + \\ &\qquad \dot{q}_{Be0}\, [\dot{q}_{Be1}^\dagger\, e_1 + \dot{q}_{Be2}^\dagger\, e_2 + \dot{q}_{Be4}^\dagger\, e_4] + \ [\dot{q}_{Be1}\, e_1 + \dot{q}_{Be2}\, e_2 + \dot{q}_{Be4}\, e_4] \dot{q}_{Be_0} \end{aligned} \tag{73}$$

Here it is evident that there must exist two massless spin one ‘Lorentz’ bosons corresponding to this ‘quantisation/gauging’ of the Lorentz symmetry. Such a particle has recently been proposed in a very interesting paper by Cahill in [60] [the coinage Lorentz boson is due to him]. The discovery of such a boson would constitute a supportive evidence for our theory. Added to the nine bosons coming from the electro-colour sector, and the three weak isospin bosons, the two Lorentz bosons make a total of fourteen. This equals the fourteen generators of G_2. has fourteen generators, so one needs four more bosons. On the other hand leaving out the $\dot{q}_{Be0}$. How two of them acquire charge we will see when we discuss complex bioctonions below. Although electric charge is being fixed in the electro-colour sector, undoubtedly, gravito-weak and electro-colour interact non-trivially in the intersection. As Furey has noted, the nilpotents ω and $\omega^\dagger$ constructed from the electro-colour ladder operators have electric charge, and also map isospin up to isospin down correctly when acting to the right on the idempotent.

Can one construct a $C\ell(6)$ algebra for the gravito-weak sector, which could be said to be independent from the electro-colour $C\ell(6)$? Without making additional assumptions / approximations, no. Under a certain approximation, yes. The approximation would consist of first making the $SU(2)$ β generators from electro-colour ladder operators, and then setting the electro-colour coupling constant to zero, which physically means that gravito-weak is a good effective symmetry under circumstances where strong and electromagnetic effects are insignificant. [We discuss the implications of this below.] The β generators were made using four octonionic directions from outside the quaternion, and two more from inside the quaternion (e_0, e_1, e_2, e_4). That leaves two

directions for making a $C\ell(2)$ to describe Lorentz symmetry, if we suitably redefine and make a quaternionic triplet from these two un-used directions. If we say label these directions as e_1 and e_4 then we can make a $C\ell(6)$ algebra from the following six ladder operators:

$$\begin{aligned} \alpha_0 &= \frac{1}{2}(ie_1 - e_2); \qquad \beta_1 = (ie_4 + ie_1\tau_3); \qquad \beta_2 = \omega^\dagger ie_1 \\ \alpha_0^\dagger &= \frac{1}{2}(ie_1 + e_2); \qquad \beta_1^\dagger = (ie_4 - ie_1\tau_3);; \qquad \beta_2^\dagger = ie_1\omega \end{aligned} \tag{74}$$

This Clifford algebra implies a non-trivial mixing between the weak sector and the Lorentz sector, because only one of the bosons coming from the Aut($\mathbb{H}$) sector has to do with gravity. Hence a connection between the gravitational force and the weak force is implied, whose experimental consequences must be explored. To our understanding, these ladder operators are different from the ones constructed by Stoica [45].

We also note another promising avenue for studying the gravi-weak symmetry, which could come from investigating the action of complex quaternions on octonions. We have recently studied the trace dynamics of what we called pure gravity case [7] [no internal symmetries]. At that time we expected weak interaction to also be part of an internal symmetry - a Yang-Mills field like the other two [electro-colour]. But now we know that the weak-interaction is a short-range spacetime symmetry, but not pertaining to 4-D spacetime; it pertains to the 8-D octonionic spacetime. Also, note that there is no dimensionless coupling constant α for the weak interaction, neither in our theory nor in the standard model. [Strong interactions and electrodynamics of course have a dimensionless coupling constant]. The weak coupling constant, i.e. the Fermi constant G_F, is dimensionful, like Newton's G_N for gravity. In fact, we have shown that our theory gives the correct value of G_F, from G_N, if we make use of the known value of mass for the Higgs [50].

The Lagrangian which we studied, thinking of it as the pure gravity case [7], and which we now re-write here allowing an adjoint of $\dot{q}_F$ to be taken, is

$$\begin{aligned} \frac{S}{C_0} = \frac{1}{2}\int \frac{d\tau}{\tau_{Pl}}\, Tr\bigg[\frac{L_P^2}{L^2c^2}\, \left(\dot{q}_B^\dagger + \beta_1\frac{L_P^2}{L^2}\dot{q}_F^\dagger\right)\, \left(\dot{q}_B + \beta_2\frac{L_P^2}{L^2}\dot{q}_F\right)\bigg] = \\ \frac{1}{2}\int \frac{d\tau}{\tau_{Pl}}\, Tr\bigg[\frac{L_P^2}{L^2}\bigg\{\dot{q}_B^\dagger\dot{q}_B + \frac{L_P^2}{L^2}\dot{q}_B^\dagger\beta_2\dot{q}_F + \frac{L_P^2}{L^2}\beta_1\dot{q}_F^\dagger\dot{q}_B + \frac{L_P^4}{L^4}\beta_1\dot{q}_F^\dagger\beta_2\dot{q}_F\bigg\}\bigg] \end{aligned} \tag{75}$$

This is precisely the same Lagrangian as our present Lagrangian (42) above, but now with the coupling constant $\alpha = 0$. Knowing that the symmetry group for this Lagrangian is the other maximal sub-group $SU(2)\times SU(2)/\mathbb{Z}_2$ of G_2, and noting that $\dot{q}_B$ is a quaternion, we can construct a $C\ell(6)$ algebra from the left minimal ideals of complex quaternions acting on complex octonions.

This will give the correct right action on the eight fermion basis S^u and again constitutes evidence for the gravito-weak symmetry. It is mediated by six bosons - the three weak isospin bosons, the two Lorentz bosons, and the Higgs boson.

The idea of a gravi-weak unification in the context of quantum field theoretic approaches has been investigated also by Nesti and Percacci [57]. They note that the complexified $SO(3,1)$ has two copies of the $SU(2)$ sub-algebra, one of which can be identified with the weak interaction, and the other with gravity. We have come to essentially the same conclusion in our analysis above, from quantum foundational considerations.

When the electro-colour and gravito-weak sector are both to be taken into account, the two Clifford algebras are not independent but have a $C\ell(4)$ overlap between them. Thus the algebra is effectively a $C\ell(8)$ algebra, which cannot be made from complex octonions. This compels us to consider the next algebra, beyond the octonions, in the Cayley-Dickson construction, namely the sedenions. However, we pursue this promising path only because because the aikyon can still be an octonion, with the complex sedenions effectively yielding three copies of the octonion algebra. This is very promising, as it can explain the three fermion generations, besides the sedenions providing a promising path towards unifying gravity with the standard model interactions. Once again, we will construct left minimal ideals from the action of complex sedenions onto themselves. We almost wholly follow the recent important work of Gillard and Gresnigt [46], interpreting it in the context of our Lagrangian.

A second promising avenue, which also fits our Lagrangian very well, is the extension beyond division algebras, to the Jordan algebra $J_3(\mathbb{O})$ of 3x3 Hermitean matrices with octonionic entries. This builds on the important and beautiful recent work of Dubois-Violette, Todorov, and their collaborators, on exceptional Jordan algebras, especially concerning the automorphism group F_4, of the algebra of $J_3(\mathbb{O})$, of which G_2 is a sub-group. We discuss both the Jordan algebra approach and the sedenion approach applied to our Lagrangian. It appears at this juncture that the Jordan algebra approach is suited to the self-adjoint part of our Lagrangian. Whereas the sedenion approach seems more relevant for the full Lagrangian, which is not self-adjoint. The anti-self-adjoint part is essential for recovery of the classical limit via spontaneous localisation. We emphasise again that this Lagrangian was not constructed to suit the investigations of division algebras, Jordan algebras, and sedenions. It was constructed to arrive at a formulation of quantum field theory which does not depend on classical time, using the methods of trace dynamics and Connes' non-commutative geometry. Hence it is very encouraging that the Lagrangian is invariant under automorphisms induced by a suitable algebra and describes well the standard model and its

unification with gravity.

4. Towards unification

We now work with the full Lagrangian of our theory, in which the total time derivative terms were not dropped, and which is given by [Eqn. (11) of [21]]

$$\begin{aligned}\mathcal{L} = Tr\Bigg[\frac{L_p^2}{L^4}\Bigg\{ i\alpha\Bigg(q_B^\dagger + \frac{L_p^2}{L^2}\beta_1 q_F^\dagger\Bigg) + L\Bigg(\dot{q}_B^\dagger + \frac{L_p^2}{L^2}\beta_1\dot{q}_F^\dagger\Bigg)\Bigg\} \\ \Bigg\{ i\alpha\Bigg(q_B + \frac{L_p^2}{L^2}\beta_2 q_F\Bigg) + L\Bigg(\dot{q}_B + \frac{L_p^2}{L^2}\beta_2\dot{q}_F\Bigg)\Bigg\}\Bigg]\end{aligned} \tag{76}$$

The Lagrangian (42) we have been using so far is a special case of this complete Lagrangian from which the total time-derivative terms have been dropped. We are going to need this full Lagrangian if we are to describe three generations of fermions - only then there are enough degrees of freedom. In fact the count of number of terms matches perfectly with what is desired. The Lagrangian (42) has 128 terms [16×8]. The full Lagrangian above has 256 terms. We are going to demand that the gravito-weak sector [64 terms] should be common amongst the three generations [because it describes space-time symmetry, not matter content]. The electro-colour sector of each generation requires 64 terms, and for three generations this comes to $64 \times 3 = 192$ terms. Add to that the 64 of gravito-weak part and we get 256. This is indeed remarkable - it matches perfectly with the number of terms in the full Lagrangian! We find this highly encouraging.

Next, we note that this full Lagrangian is a product of two terms each of which is the sum of two octonions (only one of them has the real direction in it). This strongly suggests that we can consider each bracket as a complex sedenion, and the Lagrangian is a product of two sedenions. This motivates us to relate to the recent work of Gillard and Gresnigt, and obtain the unification of the standard model [three generations] with gravity, through minimal left ideals made from the action of complex sedenions on themselves. We outline this construction below, as to what we anticipate [which is somewhat different from Gillard and Gresnigt's conclusions]. The automorphism group of the sedenions is

$$\mathrm{Aut}(\mathbb{S}) = \mathrm{Aut}(\mathbb{O}) \times S_3 \tag{77}$$

where S_3 is the permutation group. It is isomorphic to Spin(8), famous for its triality. This suggests that we could think of these automorphisms as being made from three copies of G_2 automorphisms,

two copies at a time. Such an inference then supports the idea that the three fermion generations arise for this reason, and this inference also dictates how we interpret the Clifford algebra made from sedenion automorphisms. The relation between three copies of Spin(8) and G_2 is elaborated at, for instance this link: $https://ncatlab.org/nlab/show/SO\%288\%29$ The exceptional group G_2 is the intersection of any two of the three Spin(7) sub-groups of Spin(8). Thus there are three such intersections and three copies of G_2 arise. Each copy of G_2 has a sub-group $SU(3)$ which has a sub-group $SU(2)$. We have also seen above that the $SU(2)$ is the group extension of the Aut($\mathbb{H}$) inside G_2. Put together, these features suggest the pattern of symmetry groups for three generations as shown in Figure 3. The Lorentz symmetry is the three-way intersection amongst the three generations. Then there are three copies of pair-wise interactions between any two generations, mediated by electroweak interactions. Each generation has its own $SU(3)_c$ symmetry. There are eight gluons, the photon, three weak bosons, four possible Higgs bosons [see next sub-section], *and* the newly proposed Lorentz bosons. One can possibly construct a $C\ell(14)$ Clifford algebra from the fourteen out of the fifteen imaginary one-vectors of the sedenions. It turns out that the full Lagrangian (76) is related to the exceptional Jordan algebra $J_3(\mathbb{O})$ in an important way. This helps us predict the values of the standard model parameters at the relatively low energies at which accelerator experiments are currently being carried out. Fig. 4 below depicts this unification.

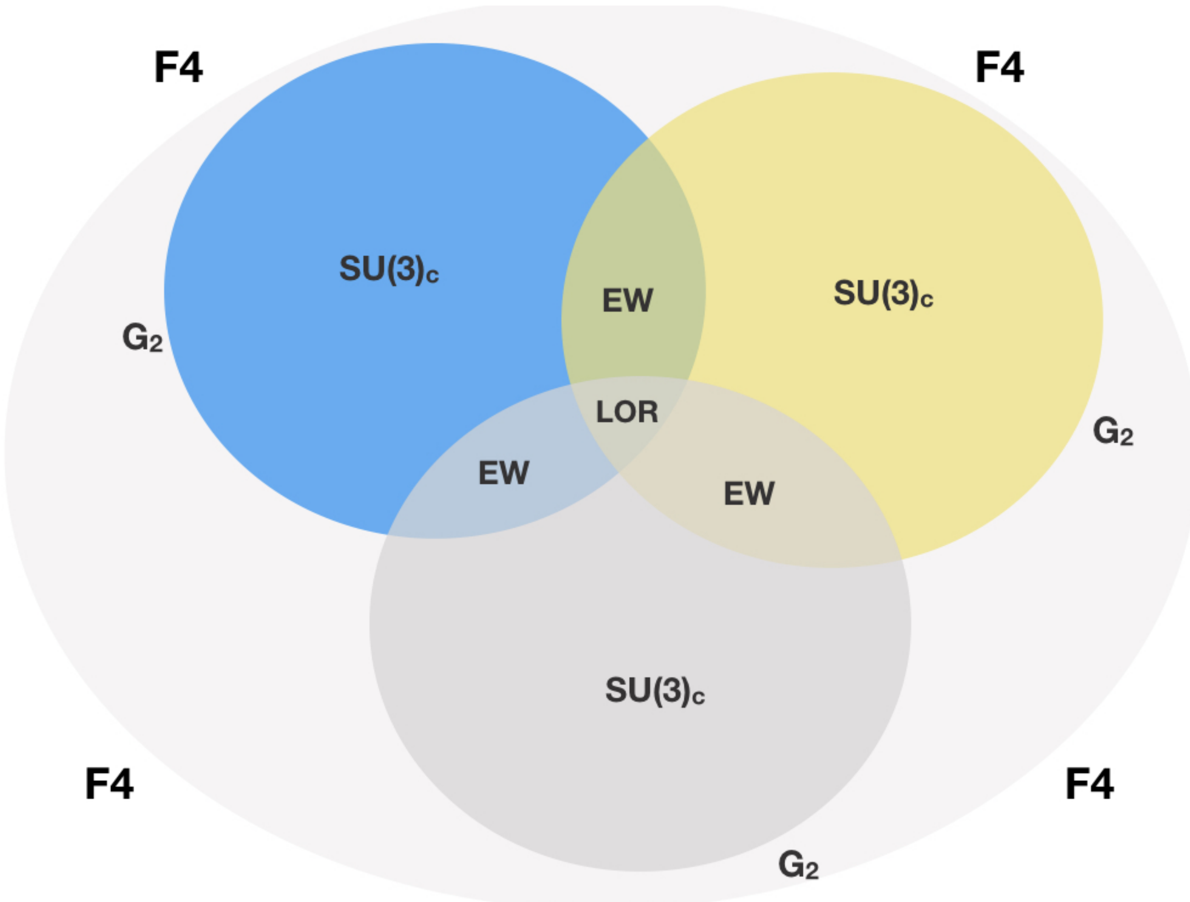
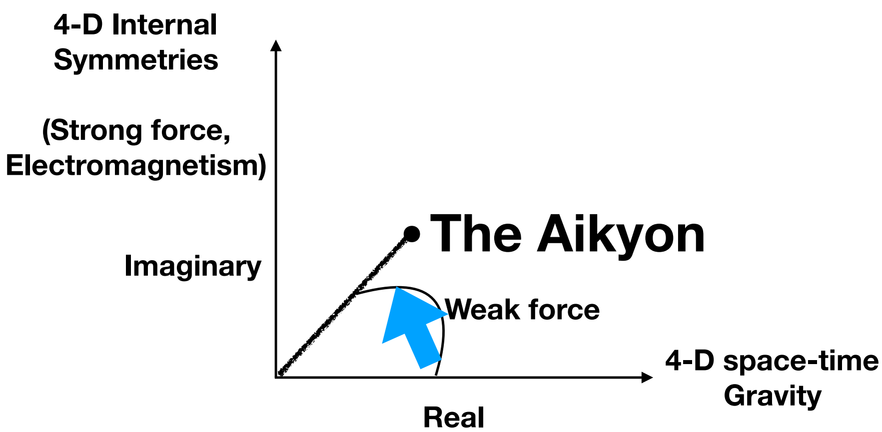


FIG. 3.

The proposed unification of the three fermion generations of the standard model, and the unification of the standard model interactions with the Lorentz interaction and hence with gravity. The symmetry group of each of the three generations is G_2 and the three copies of G_2 are all embedded in one copy of F_4 . The G_2 have a three-way intersection which is the Lorentz symmetry, and they have three pairwise intersections which is the electro-weak symmetry. Each fermion generation has its own $SU(3)_c$ symmetry. The unification symmetry group is F_4 (the automorphism group of the exceptional Jordan algebra) for the self-adjoint sector, and E_6 for the full sector. From [50]

The Aikyon Theory



$$\frac{S}{C_0} = \frac{1}{2}\int \frac{d\tau}{\tau_{Pl}}\, Tr\left[\frac{L_p^2}{L^2}\left(\dot{\tilde{Q}}_B^{\dagger} + \frac{L_p^2}{L^2}\beta_1 \dot{\tilde{Q}}_F^{\dagger}\right)\left(\dot{\tilde{Q}}_B + \frac{L_p^2}{L^2}\beta_2 \dot{\tilde{Q}}_F\right)\right]$$

FIG. 4.

The Aikyon Theory: At the Planck scale, there is no distinction between space-time symmetry and internal symmetry. Physical space is eight dimensional non-commutative octonionic space. One can imagine it as a 2-D complex plane, where the real axis represents 4-D to-be-spacetime, and the imaginary axis represents 4-D to be internal symmetries. The aikyon is an elementary particle, say an electron, *along with* the fields it produces. We do not make a distinction between the particle and the fields it produces. This is evident from the form of the action for an aikyon, shown above: variables with subscript B stand for the four known forces, and those with subscript F for any of the 24 known fermions of the three generations of the standard model. The Lagrangian is unchanged if B and F variables are interchanged. This is suggestive of super-symmetry. And since the B-variables include both gravity and gauge-fields, there is a gauge-gravity duality. The aikyon evolves in this 8-D space in Connes time. The aikyon is a 2D object, as if a membrane [2-brane]. Motion along the real axis is caused by gravity, along vertical axis by electro-colour force, and from real to imaginary by the weak force. Or we can just say, the aikyon moves in the 8D space under the influence of the unified force, given by the B-variable in the action. There is one such action term for every aikyon in this space. Different aikyons interact by ‘colliding’ with each other. The coordinates of this 8D space are the eight components of an octonion. Algebra automorphisms transform one coordinate system to another. These are the analog of general coordinate transformations of general relativity and internal gauge symmetries of gauge theories, and hence unify those concepts. The theory is invariant under 8D algebra automorphisms. And because the laws of motion are those of trace dynamics, this is already a quantum theory [50].

A. The Four Higgs bosons?

The terms T_4 and T_8 in the Lagrangian, discussed in Eqn. (61) above, have not been used up or interpreted so far. There will be one such pair of terms for each fermion generation and each pair likely constitutes one Higgs boson, which is used up to give mass to particles of that generation. In all probability, this mechanism, which remains to be worked out, works between a pair of generations, via electro-weak interaction, as suggested in Fig. 3 above. In addition, there is one additional boson generated along with the Lorentz bosons, as we saw above, via the symmetry LOR common across the three generations, as shown in Fig. 3. This could possibly be the Higgs boson observed in accelerator experiments. We hope to investigate the Higgs mechanism in our theory in the near future.

In this way, all the terms in our Lagrangian (76) have been accounted for. They describe the unification of three generations of standard model fermions and bosons, with the Lorentz interaction, and hence with gravitation. The symmetry group which describes this unification is E_6, with F_4 being the symmetry of the self-adjoint part of the Lagrangian.

IX. RECOVERING CLASSICAL SPACE-TIME, AND QUANTUM FIELD THEORY, FROM GENERALIZED TRACE DYNAMICS

A. Emergence of quantum theory below the Planck scale

We can now describe the trace dynamics equations of motion without having to refer the matrices to the octonionic coordinate system. The equations of motion can be derived in a compact way from the Lagrangian (46) which we reproduce here for easy reference [53]

$$Tr\mathcal{L} = \frac{1}{2}a_1 a_0 \; Tr\left[\dot{q}_1^\dagger \dot{q}_2 - \frac{\alpha^2 c^2}{L^2} q_1^\dagger q_2\right] \tag{78}$$

where $S \equiv \int d\tau \; Tr\mathcal{L}$ and $a_0 \equiv L_P^2/L^2$ and $a_1 \equiv \hbar/cL_P$. Also, $q_1^\dagger = q_B^\dagger + \beta_1 q_F^\dagger$ and $q_2 = q_B + \beta_2 q_F$. Variation of this Lagrangian with respect to $q_1^\dagger$ and q_2 gives the following two Euler-Lagrange equations of motion:

$$\ddot{q}_1^\dagger = -\frac{\alpha^2 c^2}{L^2} q_1^\dagger; \qquad \ddot{q}_2 = -\frac{\alpha^2 c^2}{L^2} q_2 \tag{79}$$

In terms of these two complex variables, the aikyon behaves like two independent complex-valued oscillators. However, the degrees of freedom of the aikyon couple with each other when expressed in terms of the self-adjoint variables q_B and q_F. This is because q_1 and q_2 both depend on q_B and q_F, the difference being that q_1 depends on β_1 and q_2 depends on β_2.

The trace Hamiltonian for the aikyon is

$$Tr\mathcal{H} = Tr[p_1\dot{q}_1^\dagger + p_2\dot{q}_2 - Tr\mathcal{L}] = \frac{a_1 a_0}{2} Tr\left[\frac{4}{a_1^2 a_0^2} p_1 p_2 + \frac{\alpha^2 c^2}{L^2} q_1^\dagger q_2\right] \tag{80}$$

and Hamilton's equations of motion are [53]

$$\dot{q}_1^\dagger = \frac{2}{a_1 a_0} p_2 \qquad \dot{q}_2 = \frac{2}{a_1 a_0} p_1; \qquad \dot{p}_1 = -\frac{a_1 a_0 \alpha^2 c^2}{2L^2} q_2; \qquad \dot{p}_2 = -\frac{a_1 a_0 \alpha^2 c^2}{2L^2} q_1^\dagger \tag{81}$$

Because the trace Lagrangian is invariant under global unitary transformations, it possesses a novel conserved charge, known as the Adler-Millard charge, and defined as follows:

$$\tilde{C} = \sum_i [q_{Bi}, p_{Bi}] - \sum_i \{q_{Fi}, p_{Fi}\} \tag{82}$$

which is the sum, over all bosonic degrees of freedom, of the commutator $[q_B, p_B]$ minus the sum, over all fermionic degrees of freedom, of the anti-commutator $\{q_F, p_F\}$. This charge plays an important role in the theory, and is responsible for emergence of quantum field theory at energies below the Planck scale. For this model, we have simply

$$C = [q_1, p_1] + [q_2, p_2] \tag{83}$$

If we express $q_1^\dagger$ and q_2 in terms of their self-adjoint and anti-self-adjoint parts as $q_1^\dagger = q_1^s + q_1^{as}$, $q_2 = q_2^s + q_2^{as}$, then we can use the self-adjoint and anti-self-adjoint parts as new dynamical variables [53],

$$q_1^s = \frac{1}{2}(q_1 + q_1^\dagger); \qquad q_1^{as} = \frac{1}{2}(q_1^\dagger - q_1); \qquad q_2^s = \frac{1}{2}(q_2 + q_2^\dagger); \qquad q_2^{as} = \frac{1}{2}(q_2 - q_2^\dagger) \tag{84}$$

and then the Lagrangian becomes

$$\mathcal{L} = \frac{a_1 a_0}{2} Tr\left[\frac{4}{a_1^2 a_0^2}\left\{\dot{q}_2^s \dot{q}_1^s + \dot{q}_2^{as} \dot{q}_1^{as}\right\} - \frac{\alpha^2 c^2}{L^2}\left\{q_1^s q_2^s + q_1^{as} q_2^{as}\right\} + \frac{4}{a_1^2 a_0^2}\left\{\dot{q}_2^s \dot{q}_1^{as} + \dot{q}_2^{as} \dot{q}_1^s\right\} - \frac{\alpha^2 c^2}{L^2}\left\{q_1^s q_2^{as} + q_1^{as} q_2^s\right\}\right] \tag{85}$$

In terms of these new variables, and in terms of the self-adjoint and anti-self-adjoint parts of the

momenta p^s and p^{as}, the real and imaginary parts of the trace Hamiltonian are given as follows [53]

$$TrH^s = \frac{a_1 a_0}{2} Tr\left[\frac{4}{a_1^2 a_0^2}\left\{p_1^s p_2^s + p_1^{as} p_2^{as}\right\} + \frac{\alpha^2 c^2}{L^2}\left\{q_1^s q_2^s + q_1^{as} q_2^{as}\right\}\right] \tag{86}$$

$$TrH^{as} = \frac{a_1 a_0}{2} Tr\left[\frac{4}{a_1^2 a_0^2}\left\{p_1^s p_2^{as} + p_1^{as} p_2^s\right\} + \frac{\alpha^2 c^2}{L^2}\left\{q_1^s q_2^{as} + q_1^{as} q_2^s\right\}\right] \tag{87}$$

The self-adjoint and anti-self-adjoint components of the Adler-Millard charge become

$$\begin{aligned} \tilde{C}^{as} &= [q_1^s, p_1^s] + [q_1^{as}, p_1^{as}] + [q_2^s, p_2^s] + [q_2^{as}, p_2^{as}] \\ \tilde{C}^s &= [q_1^s, p_1^{as}] + [q_1^{as}, p_1^s] + [q_2^s, p_2^{as}] + [q_2^{as}, p_2^s] \end{aligned} \tag{88}$$

We can now appreciate the significance of the anti-self-adjoint part of the Hamiltonian. It causes non-unitary evolution, equivalent to rapid variations in the Hamiltonian across the matrix dynamics phase space. The ratio L_P/L, which appears in the various terms, plays a very important role. If $L \gg L_P$, the variations are ignorable on scales much larger than Planck length, and hence can be justifiably averaged, so as to arrive at the emergent theory, which is in fact quantum theory. And indeed, on the scales at which current experiments are being conducted, L is much larger than Planck length, and the anti-self-adjoint part of the Hamiltonian can be safely neglected. We are completely justified in using the laws of quantum field theory at these scales.

In the theory of trace dynamics, at energies below Planck scale, quantum theory (without a background spacetime) emerges. This is achieved by coarse-graining the theory over many Planck time scales, and by applying the methods of statistical thermodynamics to arrive at the emergent quantum theory. This happens provided the self-adjoint part of the Adler-Millard charge can be neglected, and the anti-self-adjoint part of the Hamiltonian can be neglected. When that happens, the Adler-Millard charge gets equipartitioned over the four degrees of freedom, the equipartitioned value is identified with Planck's constant, and quantum commutation relations emerge, for the statistically averaged dynamical variables at statistical equilibrium:

$$[q_1^s, p_1^s] = i\hbar, \qquad [q_1^{as}, p_1^{as}] = i\hbar, \qquad [q_2^s, p_2^s] = i\hbar, \qquad [q_2^{as}, p_2^{as}] = i\hbar \tag{89}$$

The averaged dynamical variables obey Heisenberg equations of motion. An equivalent Schrödinger picture can also be constructed. Working in this framework, we have demonstrated the existence

of a ground state [53] in this emergent theory, which we call spontaneous quantum gravity. This ground state possibly has significant implications for the issue of singularity avoidance in quantum cosmology. The analysis reported there could also assist in working out the running of the coupling constant α as a function of the coarse-graining scale. One can from here make the transition to the q_B and q_F dynamical variables and write the quantum theory in terms of those variables. If we keep the total time derivative terms in the Lagrangian, we can analogously write the equations of motion for the three (interacting) fermion generations. The octonionic quantum mechanics will be described by the exceptional Jordan algebra.

There are circumstances though, when the anti-self-adjoint part of the Hamiltonian becomes important. This happens if the electro-colour interactions cause a sufficiently large number of aikyons to get entangled with each other. The entangled system has an associated effective length $L_{eff} \sim L/N$ where N is the number of entangled aikyons. If N is sufficiently large, the effective length goes below Planck length. As a result, we get that L_P/L_{eff} becomes larger than Planck length. Hence the approximation that the variations in the anti-self-adjoint part can be coarse-grained over to arrive at the emergent quantum theory is no longer valid. Superpositions of quantum states break down on laboratory time-scales. This is the process of spontaneous localisation which causes the classical limit to emerge. We re-emphasise that the anti-self-adjoint part was not added to the Lagrangian by hand in an ad-hoc way. Having it in the Lagrangian is essential in order to have a unified theory of interactions. This gives a fundamental origin for the Ghirardi-Rimini-Weber model of objective collapse.

B. Spontaneous localisation

In earlier papers [7, 21, 62] we have sketched preliminary ideas as to how spontaneous localisation of fermions gives rise to the emergence of a classical space-time. Here we recall those ideas, and explain what new things we learn from the present analysis, now that we know that the underlying space is octonionic. Let us try to understand which terms in the Lagrangian can contribute to the anti-self-adjoint part of the Lagrangian. We would require such terms to also have a component in the lower half 'real' quaternionic plane, for them to have a role in the emergence of 4-D space-time. The first term T_1 has a self-adjoint part along the real direction; hence it is not expected to directly contribute to spontaneous localisation. This possibly explains why gravitation is not localised; the weak interaction is short range because the associated bosons are massive. Prior to the electro-weak symmetry breaking the weak interaction too would be long range. The $U(1)$ gauge interaction is

associated with a self-adjoint operator [the number operator made from the electro-colour ladder operators]. This possibly explains why the electromagnetic interaction is not localised either. The $SU(3)_c$ gauge bosons span only a part of the lower-half plane, belonging 'mostly' to the upper-half plane, having originally been assigned to the purely imaginary directions (e_3, e_5, e_6, e_7). It remains to be investigated if the Bose-Einstein nature of the bosonic statistics is playing a role in bosons not being localised [unless they are massive]. It seems clear though that bosons by themselves cannot cause / undergo spontaneous localisation. Similarly, it is not clear to us if the terms T_4 and T_8 take part in causing spontaneous localisation. Most likely, these terms, being potentially three of the four Higgs bosons - are those three Higgs particles which are consumed to give masses to particles. Hence these terms will not be present in the low energy universe.

The role of causing spontaneous localisation then rests with the terms T_2, T_3, T_6, T_7 which describe interactions of the fermions with the bosons. Because of the fermions, these terms span the entire octonionic plane. They all have a self-adjoint part as well as an anti-self-adjoint part. Thus, they contribute to the anti-self-adjoint part of the Hamiltonian, while also having a self-adjoint part. Hence they provide an ideal set-up for collapse to take place when sufficiently many fermions get entangled with each other so as to make 'larger than Planck length scale' imaginary variations in the Hamiltonian. This localises the fermions to one or the other specific eigenvalues of the self-adjoint part of the fermionic Grassmann matrices. This is how a set of entangled fermions acquires an emergent classical position, giving rise to an emergent 4-D space-time. The space-time is 4-D because it is in fact the quaternionic subset of the F_4 symmetry of the octonions, and it has the desired local Lorentz symmetry, given by $\mathrm{Aut}(\{\mathbb{H}\})$ and the related $C\ell(2)$ which generates $SL(2, C)$. This is the part of the symmetry that is common across the three G_2 groups of symmetries of the three fermion generations. The Lorentz symmetry is already present in the Planck scale theory. It emerges as classical precisely in the same way that classical Maxwell electrodynamics emerges from quantum electrodynamics in the macroscopic limit.

How does gravitation emerge? Local Lorentz symmetry of the emergent space-time is generated by spontaneous localisation of one set of entangled fermions, to one specific eigenvalue of $\dot{q}_F/q_F$. It forces the coupled bosonic field also to a specific eigen-value of $\dot{q}_B$. A different set of localised entangled fermions gives rise to another eigenvalue of the coupled Lorentz field. And so on. That is how the Lorentz symmetry is gauged, giving rise to gravity. However, the localisation is of the squared Dirac operator, $Tr[D_B]^2$, one per aikyon. It is not the localisation of $Tr[D_B]$. Hence we infer a spin 2 gravitational field, not a spin one Lorentz field. We explain in some detail in [7] how gravitation emerges, as also the laws of classical general relativity. Though at that time we did not

realise that $\dot{q}_B$ describes a spin one massless boson, nor that we need octonions. It is clear that we must not quantise the gravitational field. Gravitation is an emergent collective phenomenon. We should quantise the spin one Lorentz field; rather, the gravito-weak field. The metric is already there in the underlying Lagrangian, in the term T_1. See also the important work of Landi and Rovelli on eigenvalues of the Dirac operator as dynamical observables of general relativity [17, 63]. See also the very interesting work of Zubkov [64] on 'Gauge theory of Lorentz group as a source of the dynamical electroweak symmetry breaking' which possibly has an intimate connection with our present analysis.

In a similar way, the coupling of fermions to the electro-colour degrees of freedom sends them to specific eigenvalues, and the collective set of eigenvalues describes the gauging of electro-colour interaction and emergence of Yang-Mills fields as internal symmetries.

Where are the other four dimensions? Classical space-time is being kept classical and 4-D by the rapid stochastic variations in the other four octonionic dimensions. The variations are stochastic because they are taking place in the aikyonic Hamiltonian of every one of the aikyons in an entangled set, in an uncorrelated manner. That is how we do not directly see the other four octonion directions. This is also the origin of the stochastic noise in collapse models, whose collapse parameters and the spectrum of the CSL noise should now be predictable from the underlying theory. This picture matches very closely with Adler's suggestion that the origin of the CSL noise is a stochastic imaginary component of the 4-D space-time metric. What we have obtained from the underlying theory is very similar to Adler's proposal [65]. In fact it is possible that the underlying octonionic space has a Poincaré symmetry, of which the 4-D Lorentz symmetry is a part, and these stochastic variations belong to the non-Lorentz part of the Poincaré group [translations, spin, torsion, and possibly a connection between strong interactions and torsion]. This also explains why the symmetry group of general relativity is the Lorentz group, and not the Poincaré group [only rotations, no translations]. Whereas the symmetry group of relativistic quantum mechanics is the Poincaré group, not the Lorentz group. Because quantum systems live in eight dimensions, not four. The other four directions, and the associated translation part of the Poincaré symmetry, are essential for a meaningful description of spin [22].

A quantum system which has not undergone collapse necessarily lives in 8-D non-commutative octonionic space, *not* in 4-D space-time. Although for most purposes the description on a 4-D spacetime background suffices and agrees with experiments done to date. However we have made predictions [e.g. the Lorentz boson, the gravito-weak unification] which are testable, and which will give evidence for the 8-D space in which quantum systems reside. Moreover, we know that

the 4-D description leads to puzzles at times - quantum non-locality, and the mysterious nature of spin. These puzzles go away in the 8-D non-commutative description. Also, there is no need for a Kaluza-Klein style spontaneous compactification of the extra four dimensions. For classical systems, spontaneous localisation effectively suppresses the extra four dimensions. Whereas quantum systems actually probe the other four dimensions - they must not be compactified. The symmetry group of the unified theory is F_4 (generalised to E_6), the group of automorphisms of the 8-D octonionic space, for three generations of fermions.

Once a classical space-time background is available, the emergent quantum theory can be related to conventional quantum field theory. We believe that under this situation our emergent theory [evolution in Connes time] coincides with the Horwitz-Stueckelberg covariant formulation of relativistic quantum mechanics [66]. When they treat the time part of spacetime also on the same footing as spacetime, and hence in a fully covariant manner, they need to introduce an extrinsic time parameter. It is plausible that this parameter is identical with Connes time.

We will now make the case that the characteristic equation of the exceptional Jordan algebra determines the free parameters of the standard model. We justify this in the next three sections by extending the standard model to the Left-Right symmetric model, and by proposing that the source charge for the right-handed sector is square-root of mass (not electric charge) and that this sector is the true precursor of gravitation, generalising the above proposal of Lorentz-weak unification to include $SU(3)_{grav}$ also in the pre-gravitation sector.

X. A THEORETICAL DERIVATION OF THE MASS-RATIOS OF THE CHARGED FERMIONS, AND THE LOW ENERGY FINE STRUCTURE CONSTANT, FROM THE EXCEPTIONAL JORDAN ALGEBRA

Describing the symmetries $SU(3)\times U(1)$ and $SU(2)\times SU(2)$ of the standard model [with Lorentz symmetry now included] requires two copies of the Clifford algebra $Cl(6,C)$ whereas the octonion algebra yields only one such independent copy. It turns out that if boundary terms are not dropped from the Lagrangian of our theory, the Lagrangian describes three fermion generations, with the symmetry group now raised to F_4 and E_6.. This admits three intersecting copies of G_2, with the $SU(2)\times SU(2)$ in the intersection, and a Clifford algebra construction based on the three copies of the octonion algebra is now possible [46]. Attention thus shifts to investigating the connection between F_4 and the three generations of the standard model.

F_4 is also the group of automorphisms of the exceptional Jordan algebra [33, 42, 43]. The

elements of the algebra are 3x3 Hermitean matrices with octonionic entries. This algebra admits an important cubic characteristic equation with real eigenvalues. Now we know that the three fermion generations differ from each other only in the mass of the corresponding fermion, whereas the electric charge remains unchanged across the generations. This motivates us to ask: if the eigenvalues of the $U(1)$ number operator constructed from the octonion algebra represent electric charge, what is represented by the eigenvalues of the exceptional Jordan algebra? Could these eigenvalues bear a relation with mass ratios of quarks and leptons? This is the question investigated now and answered in the affirmative. Using the very same octonion algebra which was used to construct a state basis for standard model fermions, we calculate these eigenvalues. Remarkably, the eigenvalues are very simple to express, and bear a simple relation with electric charge. We describe how they relate to mass ratios. In particular we find that the ratios of the eigenvalues match with the square root of the mass ratios of charged fermions. [These eigenvalues are invariant under algebra automorphisms, the automorphism group being F_4, and the automorphisms of one chosen coordinate representation of the fermions, as below, give other equivalent coordinate representations for the same set of fermions. Octonions serve as coordinate systems on the eight dimensional octonionic space-time manifold on which the elementary fermions live.]

Thus we are asking that when the octonions representing the three fermion generations are used as the off-diagonal entries in the 3x3 Jordan matrices, and the diagonal entries are the electric charges, what is the physical interpretation of the eigenvalues of the characteristic equation of $J_3(\mathbb{O})$? These eigenvalues are made from the invariants of the algebra, and hence are themselves invariants. So they are likely to carry significant information about the standard model. This is what we explore, and we argue that these eigenvalues inform us about mass-ratios of elementary particles, and about the coupling constants of the standard model.

We also propose a diagrammatic representation, based on octonions and F_4, of the fourteen gauge bosons, and the (8x2)x3 = 48 fermions of three generations of standard model, along with the four Higgs. We attempt to explain why there are not three generations of bosons, and re-express our Lagrangian in a form which explicitly reflects this fact. We also argue as to how this Lagrangian might directly lead to the characteristic equation of the exceptional Jordan algebra, and reveal why the eigenvalues might be related to mass. Furthermore, we identify the standard model coupling constants in our Lagrangian, and by relating them to the eigenvalues of $J_3(\mathbb{O})$ we provide a theoretical derivation of the asymptotic fine structure constant value 1/137.xxx

It is known that since F_4 does not have complex representations, it cannot give a representation of the fermion states. It has hence been suggested that the correct representation could come from

the next exceptional Lie group, E_6, which is the automorphism group of the complexified exceptional Jordan algebra. This aspect is currently being investigated by several researchers, including the present author. However, the standard model free parameters certainly cannot come from the characteristic equation related to E_6, because the roots of this equation are not real numbers in general. It is clear that the parameters must then come from the roots of the characteristic equation of F_4, which in a sense is the self-adjoint counterpart of the equation for E_6. It is in this spirit that the present investigation is carried out, and the results we find suggest that the present approach is indeed the correct one, as regards determining the model parameters. One must investigate E_6 for representations, but F_4 for the parameter values.

A. The Lagrangian for three generations

As we saw above, the action and Lagrangian for the three generations of standard model fermions, fourteen gauge bosons, and four potential Higgs bosons, are given by [50]

$$\frac{S}{\hbar} = \int \frac{d\tau}{t_P}\, \mathcal{L} \qquad ; \qquad \mathcal{L} = \frac{1}{2} Tr\left[\frac{L_p^2}{L^2} \dot{\tilde{Q}}_1^\dagger\, \dot{\tilde{Q}}_2\right] \tag{90}$$

Here,

$$\dot{\tilde{Q}}_1^\dagger = \dot{\tilde{Q}}_B^\dagger + \frac{L_p^2}{L^2}\beta_1 \dot{\tilde{Q}}_F^\dagger; \qquad \dot{\tilde{Q}}_2 = \dot{\tilde{Q}}_B + \frac{L_p^2}{L^2}\beta_2 \dot{\tilde{Q}}_F \tag{91}$$

and

$$\dot{\tilde{Q}}_B = \frac{1}{L}(i\alpha q_B + L\dot{q}_B); \qquad \dot{\tilde{Q}}_F = \frac{1}{L}(i\alpha q_F + L\dot{q}_F) = \tag{92}$$

By defining

$$q_1^\dagger = q_B^\dagger + \frac{L_P^2}{L^2}\beta_1 q_F^\dagger \qquad ; \qquad q_2 = q_B + \frac{L_P^2}{L^2}\beta_2 q_F \tag{93}$$

we can express the Lagrangian as

$$\begin{aligned} \mathcal{L} &= \frac{L_P^2}{2L^2}\, Tr\left[\left(\dot{q}_1^\dagger + \frac{i\alpha}{L} q_1^\dagger\right) \times \left(\dot{q}_2 + \frac{i\alpha}{L} q_2\right)\right] \\ &= \frac{L_P^2}{2L^2}\, Tr\left[\dot{q}_1^\dagger \dot{q}_2 - \frac{\alpha^2}{L^2} q_1^\dagger q_2 + \frac{i\alpha}{L} q_1^\dagger \dot{q}_2 + \frac{i\alpha}{L} \dot{q}_1^\dagger q_2\right] \end{aligned} \tag{94}$$

We now expand each of these four terms inside of the trace Lagrangian, using the definitions of q_1 and q_2 given above:

$$
\begin{aligned}
\dot{q}_1^\dagger \dot{q}_2 &= \dot{q}_B^\dagger \dot{q}_B + \frac{L_P^2}{L^2}\dot{q}_B^\dagger \beta_2 \dot{q}_F + \frac{L_P^2}{L^2}\beta_1 \dot{q}_F^\dagger \dot{q}_B + \frac{L_P^4}{L^4}\beta_1 \dot{q}_F^\dagger \beta_2 \dot{q}_F \\
q_1^\dagger q_2 &= q_B^\dagger q_B + \frac{L_P^2}{L^2} q_B^\dagger \beta_2 q_F + \frac{L_P^2}{L^2}\beta_1 q_F^\dagger q_B + \frac{L_P^4}{L^4}\beta_1 q_F^\dagger \beta_2 q_F \\
q_1^\dagger \dot{q}_2 &= q_B^\dagger \dot{q}_B + \frac{L_P^2}{L^2} q_B^\dagger \beta_2 \dot{q}_F + \frac{L_P^2}{L^2}\beta_1 q_F^\dagger \dot{q}_B + \frac{L_P^4}{L^4}\beta_1 q_F^\dagger \beta_2 \dot{q}_F \\
\dot{q}_1^\dagger q_2 &= \dot{q}_B^\dagger q_B + \frac{L_P^2}{L^2}\dot{q}_B^\dagger \beta_2 q_F + \frac{L_P^2}{L^2}\beta_1 \dot{q}_F^\dagger q_B + \frac{L_P^4}{L^4}\beta_1 \dot{q}_F^\dagger \beta_2 q_F
\end{aligned}
\tag{95}
$$

There are fourteen gauge bosons (equal to the number of generators of G_2). These are the eight gluons, the three weak isospin vector bosons, the photon, and the two Lorentz bosons. These bosons, along with one Higgs, can be accounted for by the four bosonic terms which form the first column in the above four sub-equations. The remaining twelve terms were proposed to describe three fermion generations and three Higgs, with the three generations being motivated by the triality of $SO(8)$. However, one important question which has not been addressed is: why does triality not give rise to three copies of the bosons?! In the framework of the present approach we tentatively explore the following answer. We know that the even-grade Grassmann numbers which form the entries of the bosonic matrices are made from even-number products of odd-grade (fermionic) Grassmann numbers, and the latter are in a sense more basic. Could it then be that bosonic degrees of freedom are made from fermionic degrees of freedom? If this were to be so, it could prevent the tripling of bosons, if we think of them as arising at the 'intersections' of the octonionic directions which represent fermions.

B. An octonionic diagrammatic representation for three fermion generations, and fourteen gauge bosons, and the Higgs

The seven imaginary unit octonions are used to make the Fano plane, which has seven points and seven lines [adding to fourteen elements; points and lines have equal status]. If we include the real direction [we have assumed $\dot{q}_{Be0}$ to be self-adjoint] also, we get an equivalent of a 3-D cube where the eight vertices now stand for the eight octonions, with one of them [the 'origin'] standing for the real line. As explained by Baez: "The Fano plane is the projective plane over the 2-element field Z_2. In other words, it consists of lines through the origin in the vector space Z_2^3. Since every such line contains a single nonzero element, we can also think of the Fano plane as consisting of the seven nonzero elements of Z_2^3. If we think of the origin in Z_2^3 as corresponding to 1 in $\mathbb{O}$,

we get the following picture of the octonions". This picture is Fig. 5 below, borrowed from Baez [36]. Considering points, lines and faces together, this structure has 26 elements [8+12+6 = 26].

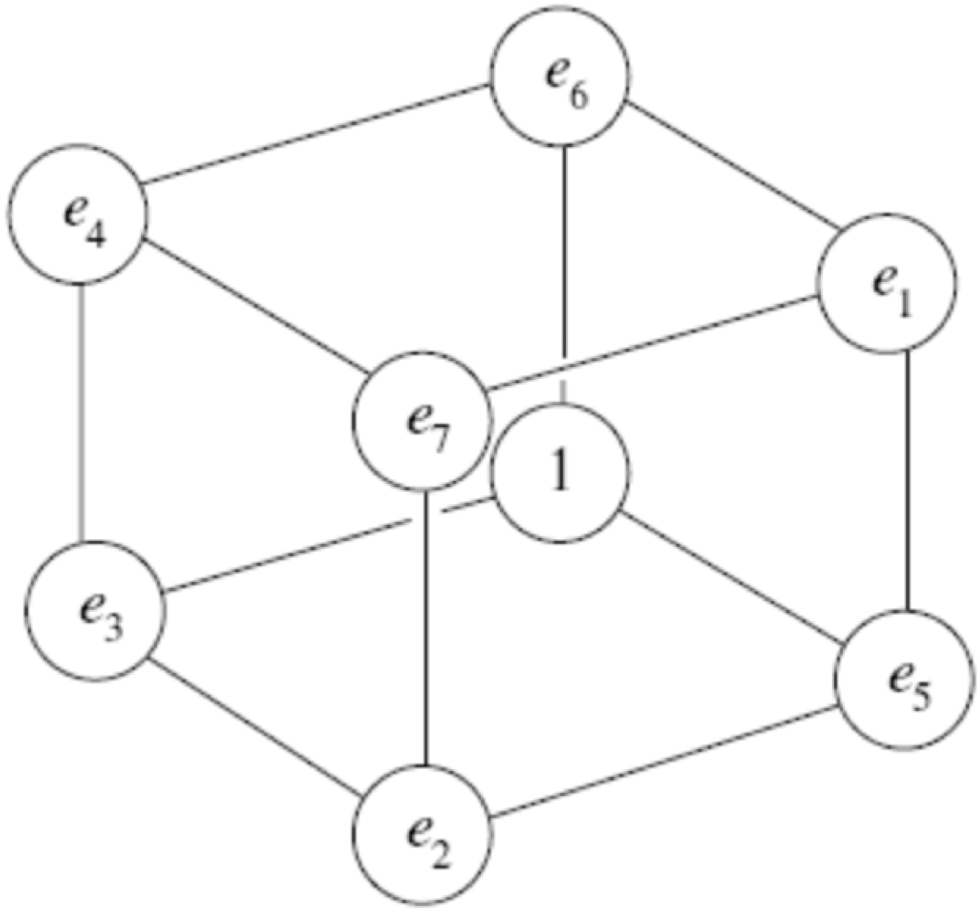


FIG. 5. The octonions [From Baez [36]].

Motivated by this representation of the octonion, and the triality of $SO(8)$, we propose the following diagrammatic representation of the standard model fermions, gauge bosons, and Higgs as shown in Fig. 6. It motivates us to think of bosons as arising as 'intersections' of the elements representing fermions. We have taken four copies of the Baez cube, with the central one at the intersection of the other three, and used them to represent the elementary particles. We now attempt to describe Fig. 6 in some detail. There is a central black-colored cube (henceforth a cube is an octonion) in the front, which represents the fourteen gauge bosons and the four Higgs bosons; we will return to this cube shortly. Then there are three more (colored) cubes: one to the left, one at the back, and one at the bottom. These are marked as Gen I, Gen II and Gen III, and represent the three fermion generations. Let us focus first on the octonion on the left, which is Gen I, and where the

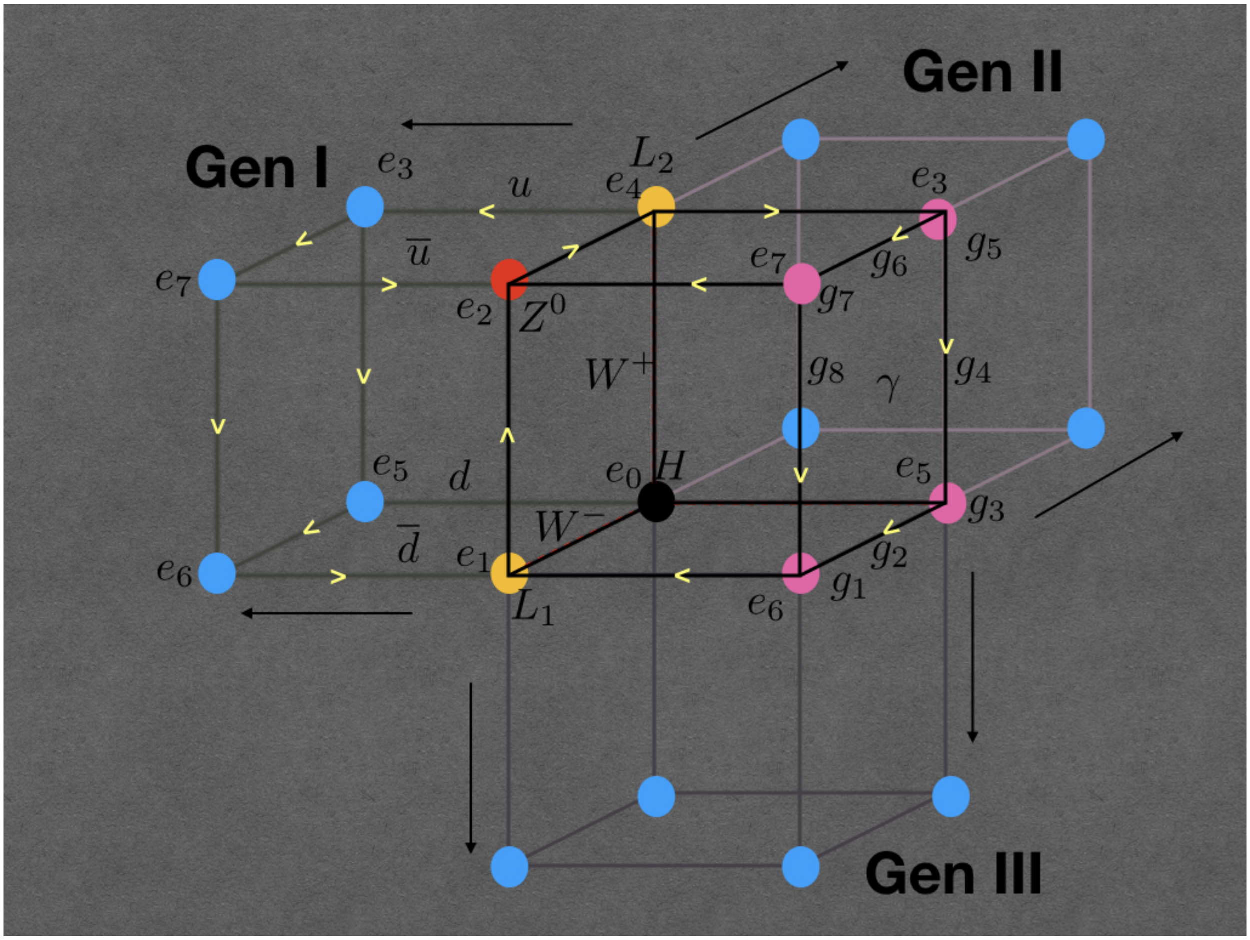


FIG. 6. The elementary particles of the standard model with three generations, represented through octonions in an F_4 diagram.. Please see text for a detailed explanation [67].

eight vertices have been marked $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ just as in the Baez cube. If e_0 were to be excluded, this cube becomes the Fano plane [Fig. 5 above] and the arrows marked in the Gen. I cube follow the same directions as in the Fano plane. In this Gen I cube, leaving out all those elements which are at the intersection with the central bosonic cube, and leaving out the face on the far left, we are left with sixteen elements: four points, eight lines, and four faces. The four points are shown in blue and are (e_3, e_5, e_6, e_7). The eight lines are: $(e_4e_3, e_7e_2, e_3e_7, e_7e_6, e_5e_6, e_6e_4, e_5e_0, e_6e_1)$. The four planes are: $(e_4e_3e_7e_2), (e_0e_5e_6e_1), (e_7e_2e_1e_6), (e_3e_4e_0e_5)$. Between them, these sixteen elements represent the eight fermions and their anti-particles in one generation, one particle / anti-particle per octonionic element.

The up quark, the down quark, and their anti-particles of one particular color are (marked by) the four lines $(e_4e_3, e_7e_2, e_0e_5, e_6e_1)$. The points (e_3, e_5, e_6, e_7) mark u, d of a second color, and the lines $(e_3e_7, e_7e_6, e_3e_5, e_5e_6)$ mark the u, d of the third color. The four planes mark the electron, the neutrino, and their anti-particles. Between them, these sixteen elements have an $SU(3)$ symmetry:

they can be correlated to the (8+8)D particle basis constructed by Furey, from the $SU(3)$ in G_2. Next, the Gen II and Gen III along with Gen I has another $SU(3)$ symmetry, which is responsible for the three generations. These three fermionic cubes represent three intersecting copies of G_2 each cube having an $SU(3)$ symmetry. The three-way intersection is $SU(2)XSU(2)$, this being the black central cube, and the bosons lie on this cube. At the same time the fermionic cubes make contact with the bosonic cube, enabling the bosons to act on the fermions.

We now try to understand the central bosonic cube. First we count the number of its elements: it gets a total of 3x10=30 elements from the three side cubes, which when added to its own 26 elements gives a total of 56. But there are a lot of common elements, so that the actual number of independent elements is much smaller, and we enumerate them now. Three points are shared two-way and three points shared three-way and the point e_0 is shared four-way; that reduces the count to 44. Nine lines are shared: three of them three way, and six of them two way, reducing the count to 32. The shared three planes reduce the count to 29. We now account for the assignment of bosons to these 29 locations.

The eight gluons are on the front right, marked by the pink points, and lines labelled g_1 to g_8, and the photon is assigned to the plane $(e_3e_7e_6e_5)$ on the front right enclosed by the gluons. The two Lorentz bosons are the yellow points e_4 and e_1 also marked L_2 and L_1. The three vector bosons are marked by the lines e_0e_1, e_0e_4 and the point e_2, also marked Z^0. The Higgs H is at the four way real point e_0. Three more Higgs are shown as follows: two planes per Higgs, e.g. the plane $e_0e_4e_2e_1$ and the mirror fermionic plane $e_3e_5e_6e_7$ on the far left in Gen I. Analogously, another Higgs is given by the bosonic plane $e_0e_1e_6e_5$ and its mirror fermionic plane at the front bottom in Gen III. The third Higgs is given by the bosonic plane $e_0e_4e_3e_5$ and its mirror fermionic plane at the back in Gen II. This way 21 elements are used up. The remaining 8 un-used elements (six lines and two planes) are assigned to eight terms in the Lagrangian representing the action of the spacetime symmetry on the gluons: these are the terms $\dot{q}_B q_B^\dagger$ and $\dot{q}_B^\dagger q_B$ in (95).

The bosonic cube lies in the intersection of the three G_2 and hence does not triplicate during the $SU(3)$ rotation which generates the three fermion generations. The symmetry group of the theory is the 52 dimensional group F_4, with 8x3=24 generators coming from the three fermionic cubes, and the rest 28 from the bosonic sector [14 + 2x3 + 8 = 28]. This diagram does suggest that one could investigate bosonic degrees of freedom as made from pairs of fermion degrees of freedom. With this tentative motivation, we return to our Lagrangian, and seek to write it explicitly as for a single generation of bosons, and three generations of fermions. Upon examination of the sub-equations in Eqn. (95) we find that the last column has terms bilinear in the fermions, and

we would like to make it appear just as the second and third column do, so that we can explicitly have three fermion generations. With this intent, we propose the following assumed definitions of the bosonic degrees of freedom, by recasting the four terms in the last column of Eqn. (95):

$$\begin{aligned}
\frac{L_P^4}{L^4}\beta_1\dot{q}_F^\dagger\beta_2\dot{q}_F &\equiv \frac{L_P^2}{L^2}\dot{q}_B\beta_2\dot{q}_F + \frac{\alpha^2}{L^2}A \\
\frac{L_P^4}{L^4}\beta_1 q_F^\dagger\beta_2 q_F &\equiv \frac{L_P^2}{L^2}q_B\beta_2 q_F + A \\
\frac{L_P^4}{L^4}\beta_1 q_F^\dagger\beta_2\dot{q}_F &\equiv \frac{L_P^2}{L^2}q_B^\dagger\beta_1\dot{q}_F^\dagger + B \\
\frac{L_P^4}{L^4}\beta_1\dot{q}_F^\dagger\beta_2 q_F &\equiv \frac{L_P^2}{L^2}\dot{q}_B^\dagger\beta_1 q_F^\dagger - B
\end{aligned} \tag{96}$$

where A and B are bosonic matrices which drop out on summing the various terms to get the full Lagrangian, With this redefinition, the sub-equations Eqn. (95) can be now written in the following form after rewriting the last column:

$$\begin{aligned}
\dot{q}_1^\dagger\dot{q}_2 &= \dot{q}_B^\dagger\dot{q}_B + \frac{L_P^2}{L^2}\dot{q}_B^\dagger\beta_2\dot{q}_F + \frac{L_P^2}{L^2}\beta_1\dot{q}_F^\dagger\dot{q}_B + \frac{L_P^2}{L^2}\dot{q}_B\beta_2\dot{q}_F \\
q_1^\dagger q_2 &= q_B^\dagger q_B + \frac{L_P^2}{L^2}q_B^\dagger\beta_2 q_F + \frac{L_P^2}{L^2}\beta_1 q_F^\dagger q_B + \frac{L_P^2}{L^2}q_B\beta_2 q_F \\
q_1^\dagger\dot{q}_2 &= q_B^\dagger\dot{q}_B + \frac{L_P^2}{L^2}q_B^\dagger\beta_2\dot{q}_F + \frac{L_P^2}{L^2}\beta_1 q_F^\dagger\dot{q}_B + \frac{L_P^2}{L^2}q_B^\dagger\beta_1\dot{q}_F^\dagger \\
\dot{q}_1^\dagger q_2 &= \dot{q}_B^\dagger q_B + \frac{L_P^2}{L^2}\dot{q}_B^\dagger\beta_2 q_F + \frac{L_P^2}{L^2}\beta_1\dot{q}_F^\dagger q_B + \frac{L_P^2}{L^2}\dot{q}_B^\dagger\beta_1 q_F^\dagger
\end{aligned} \tag{97}$$

The terms now look harmonious and we can see a structure emerging - the first column are bosonic terms and these are not triples. The remaining terms are four sets of three each [to which their adjoints will eventually get added] which can clearly describe three generations of the four sets, which is what we had in the Jordan matrices in the previous section. Putting it all together, we can now rewrite the Lagrangian so that it explicitly looks like the one for gauge bosons and four sets of three generations of fermions, as in the Jordan matrix:

$$\begin{aligned}
\mathcal{L} &= \frac{L_P^2}{2L^2}\,Tr\left[\left(\dot{q}_1^\dagger + \frac{i\alpha}{L}q_1^\dagger\right)\times\left(\dot{q}_2 + \frac{i\alpha}{L}q_2\right)\right] \\
&= \frac{L_P^2}{2L^2}\,Tr\left[\dot{q}_1^\dagger\dot{q}_2 - \frac{\alpha^2}{L^2}q_1^\dagger q_2 + \frac{i\alpha}{L}q_1^\dagger\dot{q}_2 + \frac{i\alpha}{L}\dot{q}_1^\dagger q_2\right] \\
&\equiv \frac{L_P^2}{2L^2}\,Tr\left[\mathcal{L}_{bosons} + \mathcal{L}_{set1} + \mathcal{L}_{set2} + \mathcal{L}_{set3} + \mathcal{L}_{set4}\right]
\end{aligned} \tag{98}$$

where

$$\mathcal{L}_{bosons} = \dot{q}_B^\dagger \dot{q}_B - \frac{\alpha^2}{L^2} q_B^\dagger q_B + \frac{i\alpha}{L} q_B^\dagger \dot{q}_B + \frac{i\alpha}{L} \dot{q}_B^\dagger q_B \tag{99}$$

$$\mathcal{L}_{set1} = \frac{L_P^2}{L^2} \dot{q}_B^\dagger \beta_2 \dot{q}_F + \frac{L_P^2}{L^2} \beta_1 \dot{q}_F^\dagger \dot{q}_B + \frac{L_P^2}{L^2} \dot{q}_B \beta_2 \dot{q}_F \tag{100}$$

$$\mathcal{L}_{set2} = -\frac{\alpha^2}{L^2} \left(\frac{L_P^2}{L^2} q_B^\dagger \beta_2 q_F + \frac{L_P^2}{L^2} \beta_1 q_F^\dagger q_B + \frac{L_P^2}{L^2} q_B \beta_2 q_F \right) \tag{101}$$

$$\mathcal{L}_{set3} = \frac{i\alpha}{L} \left(\frac{L_P^2}{L^2} q_B^\dagger \beta_2 \dot{q}_F + \frac{L_P^2}{L^2} \beta_1 q_F^\dagger \dot{q}_B + \frac{L_P^2}{L^2} q_B^\dagger \beta_1 \dot{q}_F^\dagger \right) \tag{102}$$

$$\mathcal{L}_{set4} = \frac{i\alpha}{L} \left(\frac{L_P^2}{L^2} \dot{q}_B^\dagger \beta_2 q_F + \frac{L_P^2}{L^2} \beta_1 \dot{q}_F^\dagger q_B + \frac{L_P^2}{L^2} \dot{q}_B^\dagger \beta_1 q_F^\dagger \right) \tag{103}$$

We see that each of these four fermionic sets could possibly be related to a Jordan matrix, after including also the adjoint part. We also see that different coupling constants appear in different sets with identical coupling in third and fourth set and no coupling in the first set. The first set could possibly describe neutrinos, charged leptons and quarks (gravitational and weak interaction), the second set charged leptons and quarks, and the third and fourth set the quarks. To establish this explicitly, equations of motion remain to be worked out and then related to the eigenvalue problem. As noted earlier, L relates to mass, and this approach could reveal how the eigenvalues of the EJA characteristic equation relate to mass. This investigation is currently in progress, and proceeds along the following lines. We take the self-adjoint part of the above Lagrangian, because that part is the one which leads to quantum field theory in the emergent approximation after coarse-graining the underlying theory. [The anti-self-adjoint part is negligible in the approximation in which quantum field theory emerges, and when it becomes significant, spontaneous localisation occurs, and classical space-time and the macroscopic universe emerges]. We vary the self-adjoint part of the Lagrangian with respect to the bosonic degree of freedom, and with respect to the three 8D-fermionic degrees of freedom, representing the three fermion generations. This yields four equations of motion, three of which are coupled matrix-valued Dirac equations for the three generations. These three coupled equations are solved by a state vector which is a three-vector

made of three 8-spinors. The eigenvalue problem for three coupled matrix equations is likely solved by the exceptional Jordan algebra, the algebra of 3x3 Hermitean matrices with octonionic entries, where the diagonal entries are identified with electric charge. That the diagonal entries are electric charge is justified by the form of the Lagrangian above, especially as written in Eqn. (45), because we see α/L as the coefficient of the potential, and its square appearing in the electrodynamics term (52) in this latest form of the Lagrangian above. This coefficient in front of the terms in Eqn. (52) then gets identified with the fine structure constant, as below.

The symmetry group associated with the self-adjoint part of the Lagrangian is F_4. The symmetry group associated with the full Lagrangian, including the anti-self-adjoint part is E_6.

We now work out the eigenvalues of the exceptional Jordan algebras for three fermion generations, demonstrate their correlation with observed mass ratios of charged fermions, and subsequently explain that this connection to mass ratios results from an extension of the standard model to the Left-Right symmetric model, with the $U(1)$ number operator associated with the right-handed sector being the square root of mass. The $U(1)$ number operator for the left-handed sector was electric charge.

C. Three fermion generations, and physical eigenvalues from the characteristic equation of the Exceptional Jordan Algebra

The exceptional Jordan algebra [EJA] $J_3(\mathbb{O})$ is the algebra of 3x3 Hermitean matrices with octonionic entries [34, 43, 48, 68]

$$X(\xi, x) = \begin{bmatrix} \xi_1 & x_3 & \tilde{x}_2 \\ \tilde{x}_3 & \xi_2 & x_1 \\ x_2 & \tilde{x}_1 & \xi_3 \end{bmatrix} \tag{104}$$

It satisfies the characteristic equation [34, 48, 68]

$$X^3 - Tr(X)X^2 + S(X)X - Det(X) = 0; \qquad Tr(X) = \xi_1 + \xi_2 + \xi_3 \tag{105}$$

which is also satisfied by the eigenvalues λ of this matrix

$$\lambda^3 - Tr(X)\lambda^2 + S(X)\lambda - Det(X) = 0 \tag{106}$$

Here the determinant is

$$Det(X) = \xi_1\xi_2\xi_3 + 2Re(x_1x_2x_3) - \sum_1^3 \xi_i x_i \tilde{x}_i \tag{107}$$

and $S(X)$ is given by

$$S(X) = \xi_1\xi_2 - x_3\tilde{x}_3 + \xi_2\xi_3 - x_1\tilde{x}_1 + \xi_1\xi_3 - x_2\tilde{x}_2 \tag{108}$$

The diagonal entries are real numbers and the off-diagonal entries are (real-valued) octonions. A tilde denotes an octonionic conjugate. The automorphism group of this algebra is the exceptional Lie group F_4. Because the Jordan matrix is Hermitean, it has real eigenvalues which can be obtained by solving the above-given eigenvalue equation.

In the present analysis we suggest that these eigenvalues carry information about mass ratios of quarks and leptons of the standard model, provided we suitably employ the octonionic entries and the diagonal real elements to describe quarks and leptons of the standard model. Building on earlier work [25, 26, 45] we recently showed that the complexified Clifford algebra $Cl(6, C)$ made from the octonions acting on themselves can be used to obtain an explicit octonionic representation for a single generation of eight quarks and leptons, and their anti-particles. In a specific basis, using the neutrino as the idempotent V, this representation is as follows [25, 50]. The α are fermionic ladder operators of $Cl(6, C)$ (please see Eqn. (34) of [50]).

$$\begin{aligned}
V &= \frac{i}{2}e_7 && [V_\nu \text{ Neutrino}]\\
\alpha_1^\dagger V = \frac{1}{2}(e_5 + ie_4) \times V &= \frac{1}{4}(e_5 + ie_4) && [\mathrm{V_{ad1}}\text{ Anti} - \text{down quark}]\\
\alpha_2^\dagger V = \frac{1}{2}(e_3 + ie_1) \times V &= \frac{1}{4}(e_3 + ie_1) && [\mathrm{V_{ad2}}\text{ Anti} - \text{down quark}]\\
\alpha_3^\dagger V = \frac{1}{2}(e_6 + ie_2) \times V &= \frac{1}{4}(e_6 + ie_2) && [\mathrm{V_{ad3}}\text{ Anti} - \text{down quark}]\\
\alpha_3^\dagger\alpha_2^\dagger V &= \frac{1}{4}(e_4 + ie_5) && [\mathrm{V_{u1}}\text{ Up quark}]\\
\alpha_1^\dagger\alpha_3^\dagger V &== \frac{1}{4}(e_1 + ie_3) && [\mathrm{V_{u2}}\text{ Up quark}]\\
\alpha_2^\dagger\alpha_1^\dagger V &= \frac{1}{4}(e_2 + ie_6) && [\mathrm{V_{u3}}\text{ Up quark}]\\
\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger V &= -\frac{1}{4}(i + e_7) && [\mathrm{V_{e+}}\text{ Positron}]
\end{aligned} \tag{109}$$

The anti-particles are obtained from the above representation by complex conjugation [25].

Note that the idempotent above, i.e. the neutrino, is $ie_7/2$ not $(1 + ie_7)/2$. The latter is for

a Dirac neutrino and the former for a Majorana neutrino. We found that identification of the neutrino with the idempotent $V = (1 + ie_7)/2$ does not give the desired values for mass-ratios and coupling constants reported here [69]. We hence propose the Majorana particle interpretation for the neutrino, and identify the neutrino with $(V - V_{cc})/2$ where V_{cc} is the complex conjugate of V. Hence the neutrino is $[(1 + ie_7) - (1 - ie_7)]/4 = ie_7/2$. Our results here demonstrate that the neutrino is a Majorana particle, and not a Dirac particle.

In the context of the projective geometry of the octonionic projective plane $\mathbb{O}P^2$ it has been shown by Baez [36] that upto automorphisms, projections in EJA take one of the following four forms, having the respective invariant trace $0, 1, 2, 3$.

$$p_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{110}$$

$$p_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{111}$$

$$p_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{112}$$

$$p_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{113}$$

Since it has earlier been shown by Furey [25] that electric charge is defined in the division algebra framework as one-third of the eigenvalue of a $U(1)$ number operator made from the generators of the $SU(3)$ in G_2, we propose to identify the trace of the Jordan matrix with the sum of the charges of the three identically charged fermions across the three generations. Thus the trace zero Jordan matrix will have diagonal entries zero, and will represent the (neutrino, muon neutrino,

tau-neutrino). The trace one Jordan matrix will have diagonal entries $(1/3, 1/3, 1/3)$ and will represent the (anti-down quark, anti-strange quark, anti-bottom quark). [Color is not relevant for determination of mass eigenvalues, and hence effectively we have four fermions per generation: two leptons and two quarks, after suppressing color]. The trace two Jordan matrix will have entries $(2/3, 2/3, 2/3)$ and will represent the (up quark, charm, top). Lastly, the trace three Jordan matrix will have entries (1, 1, 1) and will represent (positron, anti-muon, anti-tau-lepton).

We have thus identified the diagonal real entries of the four Jordan matrices whose eigenvalues we seek. We must next specify the octonionic entries in each of the four Jordan matrices. Note however that the above representation of the fermions of one generation is using complex octonions, whereas the entries in the Jordan matrices are real octonions. So we devise the following scheme for a one-to-one map from the complex octonion to a real octonion. Since we are ignoring color, we pick one out of the three up quarks, say $(e_4 + ie_5)$, and one of three anti-down quarks, say $(e_5 + ie_4)$. Since the representation for the electron and the neutrino use e_7 and a complex number, it follows that the four octonions we have picked form the quaternionic triplet (e_4, e_5, e_7) [we use the Fano plane convention shown in Fig. 7 below]. Hence the four said octonions are in fact complex quaternions, thus belonging to the general form

$$(a_0 + ia_1) + (a_2 + ia_3)e_4 + (a_4 + ia_5)e_5 + (a_6 + ia_7)e_7 \tag{114}$$

where the eight a-s are real numbers. By definition, we map this complex quaternion to the following real octonion:

$$a_0 + a_1e_1 + a_5e_2 + a_3e_3 + a_2e_4 + a_4e_5 + a_7e_6 + a_6e_7 \tag{115}$$

Note that the four real coefficients in the original complex quaternion have been kept in place, and their four imaginary counterparts have been moved to the octonion directions (e_1, e_2, e_3, e_6) now as real numbers. Clearly, the map is reversible, given the real octonion we can construct the equivalent complex quaternion representing the fermion. We can now use this map and construct the following four real octonions for the neutrino, anti-down quark, up quark and the positron, respectively, after comparing with their complex octonion representation above.

$$V_\nu = \frac{i}{2}e_7 \longrightarrow \frac{1}{2}e_6 \tag{116}$$

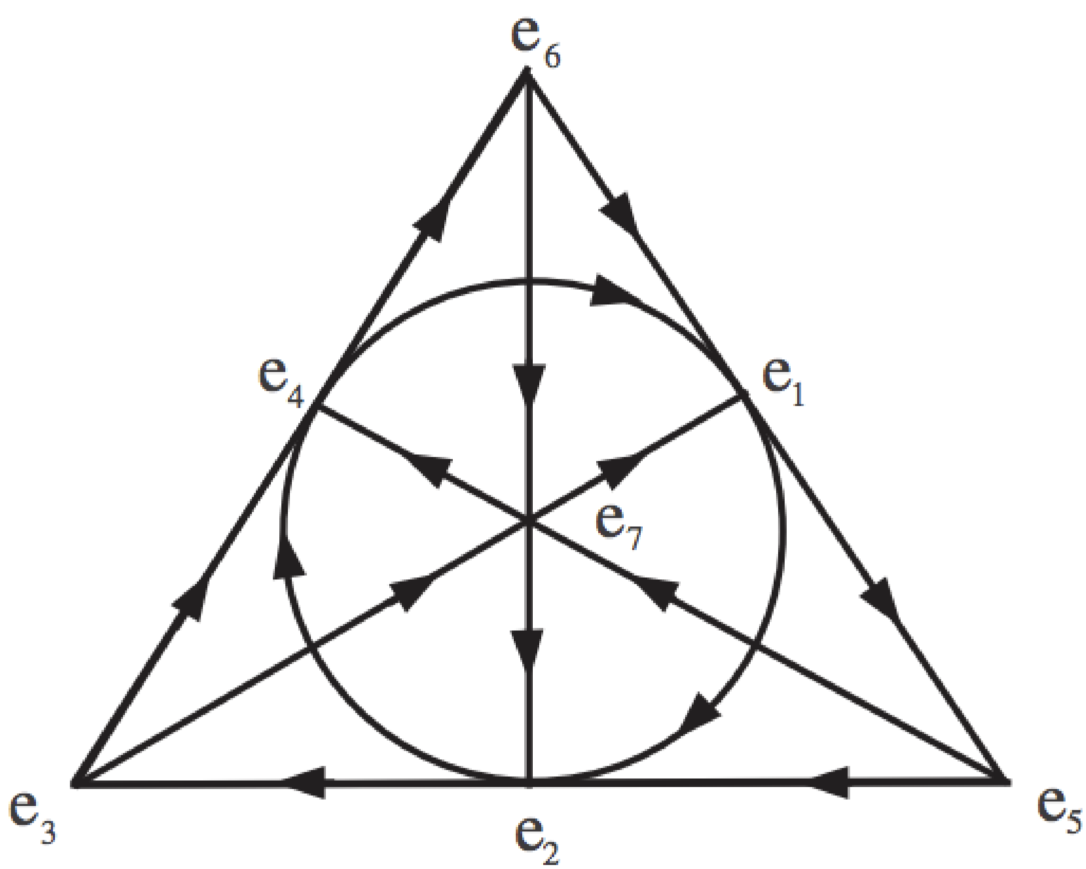


FIG. 7. The Fano plane.

$$V_{ad} = \frac{1}{4}e_5 + \frac{i}{4}e_4 \longrightarrow \frac{1}{4}e_5 + \frac{1}{4}e_3 \tag{117}$$

$$V_u = \frac{1}{4}e_4 + \frac{i}{4}e_5 \longrightarrow \frac{1}{4}e_4 + \frac{1}{4}e_2 \tag{118}$$

$$V_{e^+} = -\frac{i}{4} - \frac{1}{4}e_7 \longrightarrow -\frac{1}{4}e_1 - \frac{1}{4}e_7 \tag{119}$$

These four real octonions will go, one each, in the four different Jordan matrices whose eigenvalues we wish to calculate. Next, we need the real octonionic representations for the four fermions [color suppressed] in the second generation and the four in the third generation. We propose to build these as follows, from the real octonion representations made just above for the first generation.

Since F_4 has the inclusion $SU(3) \times SU(3)$, one $SU3)$ being for color and the other for generation, we propose to obtain the second generation by a $2\pi/3$ rotation on the first generation, and the third generation by a $2\pi/3$ rotation on the second generation. By this we mean the following construction, for the four respective Jordan matrices, as below. It is justified as follows: One of the two $SU(3)$ is color $SU(3)_c$ and has already been used up to write down the three different color states of each quark, with one pair of imaginary octonion directions fixed for a given color. The other $SU(3)$ is for generations. It is then evident from symmetry considerations that the corresponding higher generation quark of a given color can be obtained by $2\pi/3$ rotation on the first generation quark, while keeping the selected pair of octonionic directions fixed.

Up quark / Charm / Top: The up quark is $(e_4/4 + e_2/4)$ We think of this as a 'plane' and rotate this octonion by $2\pi/3$ by left multiplying it by $e^{2\pi e_4/3} = -1/2 + \sqrt{3}e_4/2$. This will be the charm quark V_c. Then we left multiply the charm quark by $e^{2\pi e_4/3}$ to get the top quark V_t. Hence we have,

$$V_c = (-1/2+\sqrt{3}e_4/2) \times V_u = (-1/2+\sqrt{3}e_4/2) \times \left(\frac{1}{4}e_4 + \frac{1}{4}e_2\right) = -\frac{1}{8}e_4 - \frac{1}{8}e_2 - \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8}e_1 \tag{120}$$

We have used the conventional multiplication rules for the octonions, which are reproduced below in Fig. 8, for ready reference. Similarly, we can construct the top quark by a $2\pi/3$ rotation on the charm:

$$\begin{aligned} V_t &= (-1/2+\sqrt{3}e_4/2) \times V_c = (-1/2+\sqrt{3}e_4/2) \times \left(-\frac{1}{8}e_4 - \frac{1}{8}e_2 - \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8}e_1\right) \\ &= -\frac{1}{8}e_4 - \frac{1}{8}e_2 + \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{8}e_1 \end{aligned} \tag{121}$$

Next, we construct the anti-strange V_{as} and anti-bottom V_{ab}, by left-multiplication of the anti-down quark V_{ad} by $e^{2\pi e_3/3}$.

$$\begin{aligned} V_{as} &= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}e_3\right) \times V_{ad} = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}e_3\right) \times \left(\frac{1}{4}e_5 + \frac{1}{4}e_3\right) \\ &= -\frac{1}{8}e_5 - \frac{1}{8}e_3 + \frac{\sqrt{3}}{8}e_2 - \frac{\sqrt{3}}{8} \end{aligned} \tag{122}$$

	e0	e1	e2	e4	e3	e5	e6	e7
e0	1	e1	e2	e4	e3	e5	e6	e7
e1	e1	-1	e4	-e2	e7	e6	-e5	-e3
e2	e2	-e4	-1	e1	e5	-e3	e7	-e6
e4	e4	e2	-e1	-1	-e6	e7	e3	-e5
e3	e3	-e7	-e5	e6	-1	e2	-e4	e1
e5	e5	-e6	e3	-e7	-e2	-1	e1	e4
e6	e6	e5	-e7	-e3	e4	-e1	-1	e2
e7	e7	e3	e6	e5	-e1	-e4	-e2	-1

FIG. 8. The multiplication table for two octonions. Elements in the first column on the left, left multiply elements in the top row.

$$\begin{aligned} V_{ab} &= \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}e_3\right)\left(-\frac{1}{8}e_5-\frac{1}{8}e_3+\frac{\sqrt{3}}{8}e_2-\frac{\sqrt{3}}{8}\right) \\ &= -\frac{1}{8}e_5-\frac{\sqrt{3}}{8}e_2-\frac{1}{8}e_3+\frac{\sqrt{3}}{8} \end{aligned} \tag{123}$$

Next, we construct the octonions for the anti-muon $V_{a\mu}$ and anti-tau-lepton $V_{a\tau}$ by left multiplying the positron V_{e^+} by $e^{2\pi e_1/3}$

$$\begin{aligned} V_{a\mu} &= \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}e_1\right)\times\left(-\frac{1}{4}e_1-\frac{1}{4}e_7\right) \\ &= \frac{1}{8}e_1+\frac{1}{8}e_7+\frac{\sqrt{3}}{8}+\frac{\sqrt{3}}{8}e_3 \end{aligned} \tag{124}$$

$$V_{a\tau} = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}e_1\right) \times \left(\frac{1}{8}e_1 + \frac{1}{8}e_7 + \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{8}e_3\right)$$
$$= \frac{1}{8}e_7 - \frac{\sqrt{3}}{8} + \frac{1}{8}e_1 - \frac{\sqrt{3}}{8}e_3 \tag{125}$$

Lastly, we construct the octonions $V_{\nu\mu}$ for the muon neutrino and $V_{\nu\tau}$ for the tau neutrino, by left multiplying on the electron neutrino V_ν with $e^{2\pi e_6/3}$

$$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}e_6\right) \times \frac{1}{2}e_6 = -\frac{1}{4}e_6 - \frac{\sqrt{3}}{4} \tag{126}$$

$$V_{\nu\tau} = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}e_6\right) \times \left(-\frac{1}{4}e_6 - \frac{\sqrt{3}}{4}\right) = -\frac{1}{4}e_6 + \frac{\sqrt{3}}{4} \tag{127}$$

We now have all the information needed to write down the four Jordan matrices whose eigenvalues we will calculate. Diagonal entries are electric charge, and off-diagonal entries are octonions representing the particles. Using the above results we write down these four matrices explicitly. The neutrinos of three generations

$$X_\nu = \begin{bmatrix} 0 & V_\nu & V_{\nu\mu}^* \\ V_\nu^* & 0 & V_{\nu\tau} \\ V_{\nu\mu} & V_{\nu\tau}^* & 0 \end{bmatrix} \tag{128}$$

The anti-down set of quarks of three generations [anti-down, anti-strange, anti-bottom]:

$$X_{ad} = \begin{bmatrix} \frac{1}{3} & V_{ad} & V_{as}^* \\ V_{ad}^* & \frac{1}{3} & V_{ab} \\ V_{as} & V_{ab}^* & \frac{1}{3} \end{bmatrix} \tag{129}$$

The up set of quarks for three generations [up, charm, top]

$$X_u = \begin{bmatrix} \frac{2}{3} & V_u & V_c^* \\ V_u^* & \frac{2}{3} & V_t \\ V_c & V_t^* & \frac{2}{3} \end{bmatrix} \tag{130}$$

The positively charged leptons of three generations [positron, anti-muon, anti-tau-lepton]

$$X_{e+} = \begin{bmatrix} 1 & V_{e+} & V_{a\mu}^* \\ V_{e+}^* & 1 & V_{a\tau} \\ V_{a\mu} & V_{a\tau}^* & 1 \end{bmatrix} \tag{131}$$

Next, the eigenvalue equation corresponding to each of these Jordan matrices can be written down, after using the expressions given above for calculating the determinant and the function $S(X)$. Tedious but straightforward calculations with the octonion algebra give the following four cubic equations:

Neutrinos: We get $Tr(X) = 0, S(X) = -3/4, Det(X) = 0$, and hence the cubic equation and roots

$$\lambda^3 - \frac{3}{4}\lambda = 0 \qquad ROOTS: \left(-\sqrt{2}\sqrt{\frac{3}{8}},\ 0,\ \sqrt{2}\sqrt{\frac{3}{8}}\right) \tag{132}$$

Anti-down-quark + its higher generations [anti-down, anti-strange, anti-bottom]: We get $Tr(X) = 1, S(X) = -1/24, Det(X) = -19/216$, and the following cubic equation and roots

$$\begin{gathered} \lambda^3 - \lambda^2 - \frac{1}{24}\lambda + \frac{19}{216} = 0 \\ ROOTS:\ \frac{1}{3} - \sqrt{\frac{3}{8}},\ \frac{1}{3},\ \frac{1}{3} + \sqrt{\frac{3}{8}} \end{gathered} \tag{133}$$

Up quark + its higher generations [up, charm, top]: We get $Tr(X) = 2, S(X) = 23/24, Det(X) = 5/108$ and the following cubic equation and roots:

$$\begin{gathered} \lambda^3 - 2\lambda^2 + \frac{23}{24}\lambda - \frac{5}{108} = 0 \\ ROOTS: \frac{2}{3} - \sqrt{\frac{3}{8}},\ \frac{2}{3},\ \frac{2}{3} + \sqrt{\frac{3}{8}} \end{gathered} \tag{134}$$

Positron + its higher generations [positron, anti-muon, anti-tau-lepton]: We get $Tr(X) = 3, S(X) = 3 - 3/8, Det(X) = 1 - 3/8$ and the following cubic equation and roots:

$$\begin{gathered} \lambda^3 - 3\lambda^2 + \left(3 - \frac{3}{8}\right)\lambda - \left(1 - \frac{3}{8}\right) = 0 \\ ROOTS:\ 1 - \sqrt{\frac{3}{8}},\ 1,\ 1 + \sqrt{\frac{3}{8}} \end{gathered} \tag{135}$$

As expected from the known elementary properties of cubic equations, the sum of the roots is

$Tr(X)$, their product is $Det(X)$, and the sum of their pairwise products is $S(X)$. Interestingly, this also shows that the sum of the roots is equal to the total electric charge of the three fermions under consideration in each of the respective cases. Whereas $S(X)$ and $Det(X)$ are respectively related to an invariant inner product and an invariant trilinear form constructed from the Jordan matrix, their physical interpretation in terms of fermion properties remains to be understood.

The roots exhibit a remarkable pattern. In each of the four cases, one of the three roots is equal to the corresponding electric charge, and the other two roots are placed symmetrically on both sides of the middle root, which is the one equal to the electric charge. All three roots are positive in the up quark set and in the positron set, whereas the neutrino set and anti-down quark set have one negative root each, and the neutrino also has a zero root. It is easily verified that the calculation of eigenvalues for the anti-particles yields the same set of eigenvalues, upto a sign. In other words, the Jordan eigenvalue for the anti-particle is opposite in sign to that for the particle. The roots are summarised in the table in Fig. 9 below, and we see that they are composed of the electric charge, and the octonionic magnitude associated with the respective particle. [The octonionic magnitude L_P^2/L^2 is the sum $\sum x_i x^i$ over the three identically charged fermions of three generations, which appears in Equation (5) above.] One expects these roots to relate to masses of quarks and leptons for various reasons, and principally because the automorphism group of the complexified octonions contains the 4D Lorentz group as well, and the latter we know relates to gravity. Since mass is the source of gravity, we expect the Lorentz group to be involved in an essential way in any theory which predicts masses of elementary particles. And the group F_4, besides being related to G_2, and a possible candidate for the unification of the four interactions, is also the automorphism group of the EJA. We have motivated how the four projections of the EJA relate naturally to the four generation sets of the fermions. Thus there is a strong possibility that the eigenvalues of the characteristic equation of the EJA yield information about fermion mass ratios, especially it being a cubic equation with real roots.

D. Evidence of correlation between the Jordan eigenvalues and the mass ratios of quarks and charged leptons

In the first generation, we note the positron mass to be 0.511 Mev, the up quark mass to be $2.3 \pm 0.7 \pm 0.5$ MeV, and the down quark mass to be $4.8 \pm 0.5 \pm 0.3$ MeV. The uncertainties in the two quark masses permit us to make the following proposal: the square-roots of the masses of the positron, up quark, and down quark possess the ratio 1 : 2 : 3 and hence they can be assigned the

The Jordan Eigenvalues

Neutrinos: Magntitude 3/4	$-\frac{\sqrt{3}}{2}$	**0**	$\frac{\sqrt{3}}{2}$
1/3 Quarks: Mag. 3/8	$\frac{1}{3}-\sqrt{\frac{3}{8}}$	$\frac{1}{3}$	$\frac{1}{3}+\sqrt{\frac{3}{8}}$
2/3 Quarks Mag. 3/8	$\frac{2}{3}-\sqrt{\frac{3}{8}}$	$\frac{2}{3}$	$\frac{2}{3}+\sqrt{\frac{3}{8}}$
Charged Leptons Mag. 3/8	$1-\sqrt{\frac{3}{8}}$	**1**	$1+\sqrt{\frac{3}{8}}$

These are all numbers in Base four !!

FIG. 9. The eigenvalues of the exceptional Jordan algebra for the various fermions. The eigenvalues are made from electric charge and the octonionic magnitude, and represent charge-mass of the corresponding fermion, in the pre-theory. The corresponding eigenmatrices [48] represent charge-mass eigenstates. The $SU(3)_c$ and $U(1)$ constructed from the $Cl(6)$ and the octonion algebra for one generation defines electric charge. However to define charge-mass and mass one must deal with F_4 and all three generations, not just one [67].

'square-root-mass numbers' $(1/3, 2/3, 1)$ respectively, these being in the inverse order as the ratios of their electric charge. The $e/\sqrt{m}$ ratios for the three particles then have the respective values $(3, 1, 1/3)$, whereas $e\sqrt{m}$ has the respective values $(1/3, 4/9, 1/3)$. The choice of square-root of mass as being more fundamental than mass is justified by recalling that in our approach, gravitation is derived from 'squaring' an underlying spin one Lorentz interaction [50]. It is reasonable then to assume that the spin one Lorentz interaction is sourced by $\sqrt{m}$, and to try to understand the origin of the square-root of the mass ratios, rather than origin of the mass ratios themselves.

At this stage, the above proposed quantised root-mass-ratios for the first generation are only an assumption; but we will justify it below through the L-R symmetric extension of the standard model [70], where we consider an $SU(3)$ gravi-color symmetry for gravitation, analogous to $SU(3)_{color}$

for QCD, and actually demonstrate a square-root mass ratio 1:2:3 for electron, up quark and down quark.] A justification might come from the following. The automorphism group G_2 of the octonions has the two maximal subgroups $SU(3)$ and $SO(4)$. These two groups have an intersection $U(2) \sim SU(2) \times U(1)$. The $SU(3)$ is identified with $SU(3)_c$, the $SU(2)$ with the weak symmetry, and the $U(1)$ with $U(1)_{em}$. Thus the $U(1)_{em}$ is a subset also of the maximal sub-group $SO(4)$ which led us to propose the Lorentz-Weak-Electro symmetry, and hence this $U(1)$ might also determine the said quantised root-mass-ratios $(1/3, 2/3, 1)$ for the positron, up quark, and down quark respectively. For now, we take these quantised root-mass-ratios as a working hypothesis. This implies, assuming a mass 0.511 MeV for the electron, a consequent predicted mass of 2.044 MeV for the up quark, and a predicted mass 4.599 MeV for the down quark.

Since the square-root-mass ratio of the anti-down quark has been set to unity, and predicted above to be 4.599 MeV ($= 9 \times 0.511$ MeV), we will calculate the square-root-mass ratios of the other particles with respect to the anti-down-quark, and demonstrate a correlation of these ratios with the Jordan eigenvalues. Also, since a negative Jordan eigenvalue is to be associated with minus of square-root mass, for finding the mass-ratio, we take the absolute value of the anti-down-quark eigenvalue, which is negative. Experimental mass values are taken from [71].

- Strange quark with respect to down quark:

$$\frac{1+\sqrt{3/8}}{1-\sqrt{3/8}} = 4.16; \quad \sqrt{\frac{95}{4.7}} = 4.50 \tag{136}$$

  The charge 1 eigenvalues are assigned to the down quark family [to be justified when we discuss E_6 in one of the sections below], with the largest value given to strange quark, and smallest to down quark. The theoretical prediction 4.16 lies jut outside the experimental range (4.21, 4.86) of the corresponding ratio (see Table I below).

- Bottom quark with respect to down quark:

$$\frac{1+\sqrt{3/8}}{1-\sqrt{3/8}} \times \frac{1+\sqrt{3/8}}{1} \times \frac{1+\sqrt{3/8}}{1-\sqrt{3/8}} = 28.44; \quad \sqrt{\frac{4180}{4.7}} = 29.82 \tag{137}$$

  The strange to down ratio has been squared, and multiplied by the ratio of largest eigenvalue to middle eigenvalue. The theoretical prediction 28.44 lies within the experimentally measured range (28.25, 30.97).

- Charm quark with respect to up quark

$$\frac{2/3+\sqrt{3/8}}{2/3-\sqrt{3/8}}=23.57;\quad \sqrt{\frac{1275}{2.3}}=23.55 \tag{138}$$

The largest eigenvalue is divided by the smallest eigenvalue, and the theoretical prediction of 23.57 lies within the experimental range (21.04, 26.87).

- Top quark with respect to up quark

$$\frac{2/3+\sqrt{3/8}}{2/3-\sqrt{3/8}}\times\frac{2/3}{2/3-\sqrt{3/8}}=289.26;\quad \sqrt{\frac{173210}{2.3}}=274.42 \tag{139}$$

The charm to up ratio is multiplied by the ratio of the middle to the smallest eigenvalue. The theoretical value of 289.26 lies within the experimental range (248.18, 310.07). How do these small fractions manage to generate the huge top quark mass?! The answer lies in the numerical coincidence that $2/3\approx 0.67$ is very close to $\sqrt{3/8}\approx 0.61$ so that $(2/3-\sqrt{3/8})^{-2}\approx 339$ gives a gain factor of over 300, making the top quark so heavy. We take this numerical coincidence as a serious indicator that this theory is on the right track. For we will see shortly, that when the Dirac neutrino is assumed, the $\sqrt{3/8}$ is replaced by $\sqrt{3/2}$ and the theoretical prediction for the top to up ratio goes completely wrong.

- Muon with respect to electron: The charge 1/3 eigenvalues have been assigned to the electron family.

$$\frac{1+\sqrt{3/8}}{1-\sqrt{3/8}}\times\frac{1/3+\sqrt{3/8}}{|1/3-\sqrt{3/8}|}=14.10;\quad \sqrt{206.7682830}=14.38 \tag{140}$$

The ratio of the largest to smallest eigenvalue for the electron family has been pre-multiplied by the strange to down ratio. There is about 4% deviation from the known mass ratio of the muon to the electron.

- Tau-lepton with respect to electron

$$\frac{1+\sqrt{3/8}}{1-\sqrt{3/8}}\times\frac{1/3+\sqrt{3/8}}{|1/3-\sqrt{3/8}|}\times\frac{1+\sqrt{3/8}}{1-\sqrt{3/8}}=58.64;\quad \sqrt{\frac{1776.86}{0.511}}=58.97 \tag{141}$$

The square of the strange to down ratio has been multiplied by the ratio of largest to smallest eigenvalue in the electron family. There is about 0.6% deviation from the experimentally determined ratio.

It is interesting to note the pattern in which the eigenvalues multiply to give us root-mass ratios for various generations. The root-mass ratio of electron, up quark, and down quark is $\frac{1}{3} : \frac{2}{3} : 1$. The root-mass ratio of strange quark with respect to down quark is the ratio of the maximum eigenvalue to the minimum eigenvalue for charge 1. The ratio of bottom quark with respect to down quark can be similarly obtained but with an additional factor of maximum eigen value. Similar pattern is visible for up quark family and electron family, it is important to note that we use charge 1 eigenvalues for down quark family whereas charge $\frac{1}{3}$ eigenvalues are used for electron family. The root-mass ratios are summarised in Fig. 10 We are using charge 1 eigenvalues for the down

Mass ratios: Square root of the mass of a charged fermion with respect to the down quark

Down quark	Strange quark	Bottom quark
1	$\dfrac{1+\sqrt{3/8}}{1-\sqrt{3/8}}$	$\dfrac{1+\sqrt{3/8}}{1-\sqrt{3/8}} \times \dfrac{1+\sqrt{3/8}}{1-\sqrt{3/8}} \times \dfrac{1+\sqrt{3/8}}{1}$
Up quark	**Charm quark**	**Top quark**
$2/3$	$\dfrac{2}{3} \times \dfrac{\frac{2}{3}+\sqrt{3/8}}{\frac{2}{3}-\sqrt{3/8}}$	$\dfrac{2}{3} \times \dfrac{\frac{2}{3}+\sqrt{3/8}}{\frac{2}{3}-\sqrt{3/8}} \times \dfrac{\frac{2}{3}}{\frac{2}{3}-\sqrt{3/8}}$
Positron	**Muon**	**Tau lepton**
$1/3$	$\dfrac{1}{3} \times \dfrac{1+\sqrt{3/8}}{1-\sqrt{3/8}} \times \dfrac{1/3+\sqrt{3/8}}{\lvert 1/3-\sqrt{3/8}\rvert}$	$\dfrac{1}{3} \times \dfrac{1+\sqrt{3/8}}{1-\sqrt{3/8}} \times \dfrac{1+\sqrt{3/8}}{1-\sqrt{3/8}} \times \dfrac{1/3+\sqrt{3/8}}{\lvert 1/3-\sqrt{3/8}\rvert}$

FIG. 10. The square-root mass numbers for charged fermions. These have the same fundamental status as quantised electric charge values 1/3, 2/3 and 1 [67].

quark family and charge $\frac{1}{3}$ eigenvalues for the electron family, this is because we are interpreting this charge as the mass number, more on this below. An interesting thing to note is that the theoretically calculated root-mass ratios are lying within the experimental range considering error for the case of quarks, and depart 4% or less for the charged leptons. Mass ratios for quarks are known more accurately from experiments than their individual masses, and we will compare against such numbers in future work.

Square root mass ratios			
Particles	Theoretical mass ratio	Minimum experimental value	Maximum experimental value
muon/electron	14.10	14.37913078	14.37913090
taun/electron	58.64	58.9660	58.9700
charm/up	23.57	21.04	26.87
top/up	289.26	248.18	310.07
strange/down	4.16	4.21	4.86
bottom/down	28.44	28.25	30.97

Table I: Comparison of theoretically predicted square-root mass ratio with experimentally known range [67].

Apart from the two mass ratios of charged leptons, other theoretical mass ratios lie within the experimental bounds [71]. On accounting for the so-called Karolyhazy correction [72] we might possibly get more accurate mass ratios for all particles including charged leptons. This will be investigated in future work.

Root-mass ratios, assuming a Dirac neutrino, and using the corresponding Jordan eigenvalues

If we had assumed the neutrino to be a Dirac fermion, we would have obtained the following mass ratios, following the same construction as for the Majorana neutrino. These are different from the above ratios, because the different idempotent $V = (1 + ie_7)/2$ for the Dirac neutrino leads to the following different Jordan eigenvalues for the fermions:

- Dirac neutrino: $-\frac{1}{2} - \frac{\sqrt{3}}{2}$, 1 , $-\frac{1}{2} + \frac{\sqrt{3}}{2}$
- Down quark family: $\frac{1}{3} - \sqrt{\frac{3}{2}},\ \frac{1}{3},\ \frac{1}{3} + \sqrt{\frac{3}{2}}$
- Up quark family: $\frac{2}{3} - \sqrt{\frac{3}{2}},\ \frac{2}{3},\ \frac{2}{3} + \sqrt{\frac{3}{2}}$
- Electron family: $1 - \sqrt{\frac{3}{2}},\ 1,\ 1 + \sqrt{\frac{3}{2}}$

For the charged fermions, the change in the roots is that $\sqrt{3/8}$ is everywhere replaced by $\sqrt{3/2}$. However for the neutrino, the change in the roots is nothing short of dramatic: the zero root has gone away, and is replaced by one, whereas the other two roots have an added factor of $-1/2$. This likely has significant implications for the origin of neutrino mass.

- Anti-strange quark with respect to anti-down quark

$$\frac{1+\sqrt{3/2}}{1-\sqrt{3/2}} = 9.89; \quad \sqrt{\frac{95}{4.7}} = 4.50 \tag{142}$$

- Anti-bottom quark with respect to anti-down quark

$$\frac{1+\sqrt{3/2}}{1-\sqrt{3/2}} \times \frac{1+\sqrt{3/2}}{1} \times \frac{1+\sqrt{3/2}}{1-\sqrt{3/2}} = 218.00; \quad \sqrt{\frac{4180}{4.7}} = 29.82 \tag{143}$$

- Charm quark with respect to up quark

$$\frac{2/3+\sqrt{3/2}}{2/3-\sqrt{3/2}} = 3.39; \quad \sqrt{\frac{1275}{2.3}} = 23.55 \tag{144}$$

- Top quark with respect to up quark

$$\frac{2/3+\sqrt{3/2}}{2/3-\sqrt{3/2}} \times \frac{2/3}{2/3-\sqrt{3/2}} = 4.05; \quad \sqrt{\frac{173210}{2.3}} = 274.42 \tag{145}$$

- Anti-muon with respect to electron

$$\frac{1+\sqrt{3/2}}{1-\sqrt{3/2}} \times \frac{1/3+\sqrt{3/2}}{|1/3-\sqrt{3/2}|} = 17.30; \quad \sqrt{206.7682830} = 14.38 \tag{146}$$

- Anti-tau lepton with respect to electron

$$\frac{1+\sqrt{3/2}}{1-\sqrt{3/2}} \times \frac{1/3+\sqrt{3/2}}{|1/3-\sqrt{3/2}|} \times \frac{1+\sqrt{3/2}}{1-\sqrt{3/2}} = 171.27; \quad \sqrt{\frac{1776.86}{0.511}} = 58.97 \tag{147}$$

It can be seen clearly that the mass ratios are way off the experimental values in this case. In our previous work on the fine structure constant as well [72, 73], it was essential to work with the Majorana neutrino, to obtain a theoretical value which agrees with experiment. The derivation of mass ratios strengthens our claim that the neutrino should be a Majorana particle and not a Dirac particle.

These ratios made from the Jordan eigenvalues suggest a possible correlation with the square-root mass ratios, and hence provide a plausible definition of a mass quantum number for standard model fermions. This definition is completely independent of trace dynamics and its Lagrangian, and is a property exclusively of the octonionic algebra. This is completely analogous to the fact that in the octonionic approach to the standard model, quantisation of electric charge is deduced

from eigenvalues of the $U(1)_{em}$ operator made from the Clifford algebra $Cl(6)$. Hence, square-root of mass is treated on the same footing as electric charge: their quantisation is a property of the algebra, not of the dynamics. The difference between charge quantisation and mass quantisation is that for finding the mass eigenstates, all three generations must be considered together, not one at a time.

The pattern that the roots and ratios follow is shown in Fig. 11. Possibly, it is more reasonable to think of logarithmic square-root mass ratios, which converts products to sums. We will see evidence for logarithmic behaviour of coupling constants when we discuss L-R symmetry.

Mass ratios: Square root of the mass of a charged fermion with respect to the down quark

Down quark 1	Strange quark $\frac{1+\sqrt{3/8}}{1-\sqrt{3/8}}$	Bottom quark $\frac{(1+\sqrt{3/8})^3}{(1-\sqrt{3/8})^2}$
Up quark $2/3$	Charm quark $\frac{2}{3} \times \frac{\frac{2}{3}+\sqrt{3/8}}{\frac{2}{3}-\sqrt{3/8}}$	Top quark $\frac{4}{9} \times \frac{\frac{2}{3}+\sqrt{3/8}}{\left(\frac{2}{3}-\sqrt{3/8}\right)^2}$
Positron $1/3$	Muon $\frac{1}{3} \times \frac{1+\sqrt{3/8}}{1-\sqrt{3/8}} \times \frac{1/3+\sqrt{3/8}}{\lvert 1/3-\sqrt{3/8}\rvert}$	Tau lepton $\frac{1}{3} \times \frac{(1+\sqrt{3/8})^2 \times (1/3+\sqrt{3/8})}{(1-\sqrt{3/8})^2 \times \lvert 1/3-\sqrt{3/8}\rvert}$

Pattern of Ratios in a Row, relative to the entry in the first column of the row.

In third row, the strange/down quark ratio is incorporated in Second and third columns, as A pre-factor.

1	$(Largest/Smallest)$	$(Largest/Smallest)^2 \times (Largest/Middle)$
2/3	$(Largest/Smallest)$	$(Largest/Smallest)^2 \times (Middle/Largest)$
1/3	$(Largest/Smallest)$	$(Largest/Smallest)^2 \times (Smallest/Largest)$

FIG. 11. The square-root mass numbers for charged fermions and the pattern that relates them to the roots.

E. The Jordan eigenvalues and the low energy limiting value of the fine structure constant

If we examine the Lagrangian term for the charged leptons in Eqn. (101), the dimensionless coupiing constant C in front of it is (upto a sign):

$$C \equiv \alpha^2 \frac{L_P^4}{L^4} \tag{148}$$

[The operator terms of the form $q_B q_F$ etc. in (101) have been correspondingly made dimensionless by dividing by L_P^2]. We assume that $\ln\alpha$ is linearly proportional to the electric charge, and that the proportionality constant is the Jordan eigenvalue corresponding to the anti-down quark (more on this when we discuss L-R symmetry breaking). The electric charge 1/3 of the anti-down quark seems to be the right choice for determining α, it being the smallest non-zero value [and hence possibly the fundamental value] of the electric charge, and also because the constant α appears as the coupling in front of the supposed quark terms in the Lagrangian, as in Eqns. (102) and (103). We hence define α by

$$\ln\alpha \equiv \lambda_{ad}\, q_{ad} = \left[\frac{1}{3} - \sqrt{\frac{3}{8}}\right] \times \frac{1}{3} \qquad \Longrightarrow \qquad \alpha^2 \approx= 0.83025195149 \tag{149}$$

where λ_{ad} is the Jordan eigenvalue corresponding to the anti-down quark, as given by Eqn. (133) and q_{ad} is the electric charge of the anti-down quark (=1/3). In order to arrive at this relation for α, we asked in what way α could vary with q, if it was allowed to vary? We then made the assumption that $d\alpha/dq \propto \alpha$. In the resulting linear dependence of $\ln\alpha$ on q, we froze the value of α at that given by the smallest non-zero charge value 1/3, taking the proportionality constant to be the corresponding Jordan eigenvalue. This dependence also justifies that had we fixed α from the zero charge of the neutrino, α would have been one, as it in fact is, in our Lagrangian. We are investigating if this way of constructing α can be further justified from the Lagrangian dynamics.

As for the value of L_P/L, we identify it with one-half of that part of the Jordan eigenvalue which modifies the contribution coming from the electric charge. [For an explanation of the origin of the factor of one-half, see the next paragraph]. Thus from the eigenvalues found above, we deduce that for neutrinos, quarks and charged leptons, the quantity L_P^2/L^2 takes the respective values $(3/16, 3/32, 3/32)$. These values are equal to one-fourth of the respective octonionic magnitudes. Thus the coupling constant C defined above can now be calculated, with α^2 as given above, and $L_P^2/L^2 = 3/32$. Furthermore, since the electric charge q, the way it is conventionally defined,

has dimensions such that q^2 has dimensions (Energy $\times$ Length), we measure q^2 in Planck units $E_{Pl} \times L_P = \hbar c$. We hence define the fine structure constant by $C = \alpha^2 L_P^4/L^4 \equiv e^2/\hbar c$, where e is the electric charge of electron / muon / tau-lepton in conventional units. We hence get the value of the fine structure constant to be

$$C = \alpha^2 L_P^4/L^4 \equiv e^2/\hbar c = \exp\left[\left[\frac{1}{3} - \sqrt{\frac{3}{8}}\right] \times \frac{2}{3}\right] \times \frac{9}{1024} \approx 0.00729713 = \frac{1}{137.04006} \tag{150}$$

The CODATA 2018 value of the fine structure constant is

$$0.0072973525693(11) = 1/137.035999084(21) \tag{151}$$

Our calculated value differs from the measured value in the seventh decimal place. In the next section, we show how incorporating the Karolyhazy length correction gives an exact match with the CODATA 2018 value, if we assume a specific value for the electro-weak symmetry breaking energy scale.

Why did we identify L_P/L with one-half of the octonionic magnitude $\sqrt{3/8}$ rather than with the magnitude $\sqrt{3/8}$ itself? The answer lies in the physical interpretation originally assigned to the length scale L. [Please see the discussion below Eqn. (69) of [7]]. The length L for an object of mass m is interpreted as the Schwarzschild radius $2Gm/c^2$ of an object of mass m, so that $L_P/L = L_P\ c^2/2Gm$, which is one-half the Compton wave-length (in Planck units) and not the Compton wavelength itself. Assuming that the octonionic magnitude has to be identified with Compton wavelength (in units of Planck length), it hence has to be divided by one-half, before equating it to L_P/L. This justifies taking $L_P^2/L^2 = 1/4 \times 3/8 = 3/32$.

Once a theoretical derivation of the asymptotic fine structure constant is known, one can write the electric charge e as

$$e = (3/32)\exp[1/9 - 1/\sqrt{24}]\ (\hbar L_P/t_P)^{1/2} \tag{152}$$

where L_P and t_P are Planck length and Planck time respectively - obviously their ratio is the speed of light. In our theory, there are only three fundamental dimensionful quantities: Planck length, Planck time, and a constant with dimensions of action, which in the emergent quantum theory is identified with Planck's constant $\hbar$. We now see that electric charge is not independent of these three fundamental dimensionful constants. It follows from them. Planck mass is also constructed from these three, and electron mass will be expressed in terms of Planck mass, if only we could

understand why the electron is some 10^{22} times lighter than Planck mass. Such a small number cannot come from the octonion algebra. In all likelihood, the cosmological expansion up until the electroweak symmetry breaking is playing a role here.

Thus electric charge and mass can both be expressed in terms of Planck's constant, Planck length and Planck time. This encourages us to think of electromagnetism, as well the other internal symmetries, entirely in geometric terms. This geometry is dictated by the F_4 symmetry of the exceptional Jordan algebra.

We have not addressed the question as to how these discrete order one eigenvalues might relate to actual low values of fermion masses, which are much lower than Planck mass. We speculatively suggest the following scenario, which needs to be explored further. The universe is eight-dimensional, not four. The other four internal dimensions are not compactified; rather the universe is very 'thin' in those dimensions but they are expanding as well. There are reasons having to do with the so-called Karolyhazy uncertainty relation [74], because of which the universe expands in the internal dimensions at one-third the rate, on the logarithmic scale, compared to our 3D space. That is, if the 4D scale factor is $a(\tau)$, the internal scale factor is $a_{int}^{1/3}(\tau)$, in Planck length units. Taking the size of the observed universe to be about 10^{61} Planck units, the internal dimensions have a width approximately 10^{20} Planck units, which is about 10^{-13} cm, thus being in the quantum domain. Classical systems have an internal dimension width much smaller than Planck length, and hence they effectively stay in [and appear to live in] four dimensional space-time. Quantum systems probe all eight dimensions, and hence live in an octonionic universe.

The universe began in a unified phase, via an inflationary 8D expansion possibly resulting as the aftermath of a huge spontaneous localisation event in a 'sea of atoms of space-time-matter' [7]. The mass values are set, presumably in Planck scale, at order one values dictated by the eigenvalues reported in the present paper. Cosmic inflation scales down these mass values at the rate $a^{1/3}(\tau)$, where $a(\tau)$ is the 4D expansion rate. Inflation ends after about sixty e-folds, because seeding of classical structures breaks the color-elctro-weak-Lorentz symmetry, and classical spacetime emerges as a broken Lorentz symmetry. The electro-weak symmetry breaking is actually a electro-weakLorentz symmetry breaking, which is responsible for the emergence of gravity, weak interaction being its short distance limit. There is no reheating after inflation; rather inflation resets the Planck scale in the vicinity of the electro-weak scale, and the observed low fermion mass values result. The electro-weak symmetry breaking is mediated by the Lorentz symmetry, in a manner consistent with the conventional Higgs mechanism. It is not clear why inflation should end specifically at the electro-weak scale: this is likely dictated by when spontaneous

localisation becomes significant enough for classical spacetime to emerge. It is a competition between the strength of the electro-colour interaction which attempts to bind the fermions, and the inflationary expansion which opposes this binding. Eventually, the expanding universe cools enough for spontaneous localisation to win, so that the Lorentz symmetry is broken. It remains to prove from first principles that this happens at around the electro-weak scale and also to investigate the possibly important role that Planck mass primordial black holes might play in the emergence of classical spacetime. I would like to thank Roberto Onofrio for correspondence which has influenced these ideas. See also [75].

If we assume that the $e/\sqrt{m}$ ratios for the first generation of the charged fermions are absolute values [valid prior to the enormous scaling down of mass] then we can assign a root-mass number $e/3$ to the positron [and hence a mass number $e^2/9$], where the electric charge e is as given in Eqn. (153). Hence the mass-number for the positron/electron is

$$\sqrt{G_N}\, m_{e+} = (1/1024)\exp[2/9 - 1/\sqrt{6}]\ (\hbar L_P/t_P)^{1/2} \tag{153}$$

where G_N is Newton's gravitational constant. Thus the mass number of the electron is $1/(137\times 9)$ of Planck mass and has to be scaled down by the factor $f = 2\times 10^{19}$ before it acquires the observed mass of 0.5 MeV. This then is also the universal factor by which the assigned mass number of every quark and charged lepton must be scaled down to get it to its current value. This is not far from the twenty orders of mass-scale-down by the Karolyhazy effect in cosmology, proposed earlier in this section. The initial ratio of the electrostatic to gravitational attraction between an electron and a positron is $e^2/(e^4/81) \sim 137 \times 81 \sim 10^4$.

F. The Karolyhazy correction to the asymptotic value of fine structure constant

In accordance with the Karolyhazy uncertainty relation (Eqn. (9) of [74]) a measured length l has a 'quantum gravitational' correction Δl given by

$$(\Delta l)^3 = L_P^2\, l \tag{154}$$

For the purpose of the present discussion we shall assume an equality sign here, i.e. that the numerical constant of proportionality between the two sides of the equation is unity. And, for the

sake of the present application to the fine structure constant, we rewrite this relation as

$$\delta \equiv \frac{L_P}{\Delta l} = \left(\frac{L_P}{l}\right)^{1/3} \tag{155}$$

We set $l \equiv l_f$ where l_f is the length scale ($\approx 10^{-16}$ cm) associated with electro-weak symmetry breaking, where classical space-time emerges from the prespacetime, prequantum theory. The assumption being that when the universe evolves from the Planck scale to the electro-weak scale [while remaining in the unbroken symmetry phase], the inverse of the octonionic length associated with the charged leptons (this being $\sqrt{3/32}$) is reset, because of the Karolyhazy correction, to

$$\sqrt{\frac{3}{32}} \qquad \longrightarrow \sqrt{\frac{3}{32}} + \delta_f \equiv \sqrt{\frac{3}{32}} + \left(\frac{L_P}{l_f}\right)^{1/3} \tag{156}$$

We can also infer this corrected length as the four-dimensional space-time measure of the length, which differs from the eight dimensional octonionic value $\sqrt{3/32}$ by the amount δ_f. If we take l_f to be 10^{-16} cm, the correction δ_f is of the order 2×10^{-6}. The correction to the asymptotic value (150) of the fine structure constant is then

$$C = \alpha^2 L_P^4 / L^4 \equiv e^2/\hbar c = \alpha^2 \left[\sqrt{\frac{3}{32}} + \left(\frac{L_P}{l_f}\right)^{1/3}\right]^4 \tag{157}$$

For $l_f = 10^{-16}$ cm $= 198$ GeV^{-1}, we get the corrected value of the fine structure constant to be 0.00729737649, which overshoots the measured CODATA 2018 value at the eighth decimal place. The electroweak scale is generally assumed to lie in the range 100 - 1000 GeV. The value $l_f = 1.3699526 \times 10^{-16}$ cm $= 144.530543605$ GeV^{-1} reproduces the CODATA 2018 value 0.0072973525693 of the asymptotic fine structure constant. The choice $l_f^{-1} = 246$ GeV gives the value 0.00729739452, whereas the choice $l_f^{-1} = 159.5 \pm 1.5$ GeV gives the range (0.00729736049, 0.00729735908). 100 GeV gives the value 0.00729732757 which is smaller than the measured value. 1000 GeV gives 0.00729754842. Thus in the entire 100 - 1000 GeV range, the derived constant agrees with the measured value at least to the sixth decimal place, which is reassuring. The purpose of the present exercise is to show that the Karolyhazy correction leads to a correction to the asymptotic value of the fine structure constant which is in the desired range - a striking fact by itself. In principle, our theory should predict the precise value of the electroweak symmetry breaking scale. Since that analysis has not yet been carried out, we predict that the ColorElectro-WeakLorentz symmetry breaking scale is 144.something GeV, because only then the theoretically

calculated value of the asymptotic fine structure constant matches the experimentally measured value.

The above discussion of the asymptotic low energy value of the fine structure constant should not be confused with the running of the constant with energy. Once we recover classical spacetime and quantum field theory from our theory, after the ColorElectro-WeakLorentz symmetry breaking, conventional RG arguments apply, and the running of couplings with energy is to be worked out as is done conventionally. Such an analysis of running couplings will however be valid only up until the broken symmnetry is restored - it is not applicable in the prespacetime prequantum phase. In this sense, our theory is different from GUTs. Once there is unification, Lorentz symmetry is unified with internal symmetries - the exact energy scale at which that happens remains to be worked out.

How then does the Planck scale prespacetime, prequantum theory know about the low energy asymptotic value of the fine structure constant? The answer to this question lies in the Lagrangian given in (98) and in particular the Lagrangian term (101) for the charged leptons. In determining the asymptotic fine structure constant from here, we have neglected the modification to the coupling that will come from the presence of q_B and q_F. This is analogous to examining the asymptotic, flat spacetime limit of a spacetime geometry due to a source - gravity is evident close to the source, but hardly so, far from it. Similarly, there is a Minkowski-flat analog of the octonionic space, wherein the effect of q_B and q_F (which in effect 'curve' the octonionic space) is ignorable, and the asymptotic fine structure can be computed. The significance of the non-commutative, non-associative octonion algebra and the Jordan eigenvalues lies in that they already determine the coupling constants, including their asymptotic values. This is a property of the algebra, even though the interpretation of a particular constant as the fine structure constant comes from the dynamics, i.e. the Lagrangian, as it should, on physical grounds.

We now justify as to why the exceptional Jordan algebra yields mass ratios, by extending the standard model to the Left-Right symmetric model, with the right sector bringing in the (precursor of) gravitation.

XI. COMPLEX SPLIT BIQUATERNIONS AND BIOCTONIONS: THE LEFT-RIGHT SYMMETRIC EXTENSION OF THE STANDARD MODEL

The quaternions were initially introduced by Hamilton to explain rotations in three dimensions, and they form a non-commutative division algebra. To begin with, the use of quaternions and

octonions (the next division algebra in the series) was very limited in physics partly because of their complicated multiplication rules and also because vector algebra was able to explain rotations in three-space, as an alternative to quaternions. In [44], Gunaydin and Gursey proposed the use of octonions to understand quarks. Following them since, several authors have investigated the application of division algebras and Clifford algebras in the context of particle physics [44]-[76]. In the present section, we build up on the earlier work obtaining particles from the left ideals of Clifford algebras [25–27, 44–46]. We use the Clifford algebras $Cl(3)$ and $Cl(7)$ to obtain a left-right symmetric model for fermions. The left-right symmetric model for fermions was introduced in 1975 by Senjanovic, Mohapatra, Pati, and Salam [77–80]; this model was not widely accepted back then because it predicted neutrinos to be massive. After the discovery of neutrino mass in 2002 [81, 82], the left-right symmetric model again gained attention. The L-R model accounts for neutrino masses using the see-saw mechanism [79], and proposes a right-handed neutrino with significantly higher Majorana mass. The right-handed neutrino interacts through gravitation only (hence the name sterile neutrino) and hence is a potential candidate for dark matter [83]. The right-handed neutrino is expected to interact through the right-handed analogue of weak force as proposed by the Pati-Salam model [84], but this force manifests itself only at high energies. The left-right symmetric model explains why charge-parity is conserved in strong interaction (the strong CP problem) and why the Higgs-coupling vanishes prior to the electroweak symmetry breaking [85, 86]. We discuss sterile neutrinos and the Higgs coupling in the context of division algebras. We also discuss alternative approaches to the Pati-Salam model in the context of unification, and propose the presence of gravity mediating bosons from division algebras in a pre-spacetime theory.

In their work [87–89], Trayling and Baylis propose a geometrical approach for understanding the $Cl(7)$ algebra. They propose a higher dimensional Kaluza-Klein theory with four spacetime dimensions accounting for rotations and boosts and four dimensions to explain electroweak and color sector of the standard model. All these eight dimensions are naturally present in the octonionic chains which make the $Cl(6)$ and $Cl(7)$ algebra. Trayling and Baylis also discuss right-handed sterile neutrinos and Higgs field coupling from $Cl(7)$. In the present section we show that without using the parity operator $\frac{1\pm\gamma_5}{2}$ we naturally get two sets of fermions with opposite chirality from $Cl(7)$. This happens because $Cl(3)$ and $Cl(7)$ have two irreducible pinor groups; therefore we are able to break them as a direct sum of two copies of $Cl(2)$ and $Cl(6)$ respectively.

We will also discuss about prospects of understanding gravity along with the other gauge fields through Clifford algebras related to the L-R model. It has been pointed out by Wilson that some parts of the Clifford algebra explain the standard model whereas some parts of it explain gravity

and there seems to be an overlap [76]. In this paper we briefly discuss about the mass ratios of fermions and how this overlap between gravity and particle physics is evident if we look at a lepto-quark picture prior to symmetry breaking. Wilson observes the Pati-Salam gauge group as a $Spin(6) \times Spin(4)$ group coming naturally from the Clifford algebras $Cl(0,6)$ and $Cl(3,3)$ [76]. It is interesting to note that $Cl(7)$ which gives us two copies of $Cl(6)$ is giving us left-right symmetric fermions whereas the real algebras $Cl(0,6)$ and $Cl(3,3)$ are giving us the Pati-Salam gauge group. Boyle also discusses getting the Pati-Salam gauge group from the intersection of the two maximal subgroups of the exceptional group E_6 [90]. E_6 is the complexified version of F_4 which is the automorphism group of exceptional Jordan matrices $J_3(O)$. A relation between exceptional groups and the standard model symmetries has been investigated by authors in [72, 90–94].

1. Division Algebras Revisted

A division algebra is an algebra in which every non-zero element has a multiplicative inverse. There are only four normed division algebras namely $\mathbf{R}, \mathbf{C}, \mathbf{H}$, and $\mathbf{O}$. $\mathbf{R}$ is the algebra of real numbers, $\mathbf{C}$ is the algebra of complex numbers, $\mathbf{H}$ is the algebra of quaternions, and $\mathbf{O}$ is the algebra of octonions. Quaternions are non-commutative in nature, whereas the octonions are neither commutative nor associative.

$\mathbf{H}$ is the algebra of the quaternions made up of one real unit and three imaginary units i, j, k. The quaternions have the following multiplication rule:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{158}$$

$$ij + ji = jk + kj = ki + ik = 0 \tag{159}$$

$$ij = k, jk = i, ki = j \tag{160}$$

The octonions make the non-commutative, non-associative division algebra $\mathbf{O}$. The octonions are made up of one real unit and 7 imaginary units $e_i^2 = -1$ for i = 1, 2,..., 7. Just like quaternions, the imaginary units of octonions anti-commute with each other. The multiplication of octonions is given by the following diagram in Fig. 12 known as the Fano plane. There are seven quaternionic subsets in the Fano plane, given by the three sides of the triangle, the three altitudes, and the incircle. Multiplication of points lying along a quaternionic subset in cyclic order (as per the arrow) is given by $e_i e_j = e_k$, whereas $e_j e_i = -e_k$.

It can be checked using the above multiplication laws that the octonions are not associative,

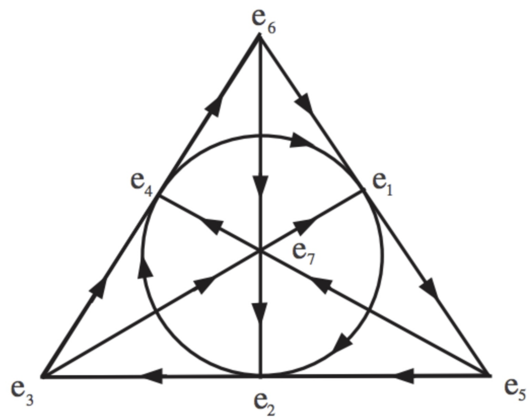


FIG. 12. The Fano plane

but we can make an associative algebra by defining octonionic chains from octonions as is shown in [25]. For more details please refer to [23, 25, 41].

2. *Clifford Algebras Revisted*

Jivet [95] and Sauter [96] showed that spinors are left ideals of matrix algebra. The mathematician Marcel Riesz showed that spinors are minimal left ideals of the Clifford algebra [97]. Several authors have shown a correspondence between the left ideals of Clifford algebras and the fermions from standard model based on Witt decomposition proposed by Ablamowicz [25–27, 45, 46, 98]. This could be done because all the fermions in the standard model are spin-half particles, and the bosons come from the symmetry groups which are also included in these algebras. The Clifford algebras are very closely related to the division algebras and were introduced as an extension of the quaternions [99].

A Clifford algebra over the field $\mathbf{R}$ is an associative algebra written as $Cl(p,q)$ such that there is a $n = p + q$ dimensional vector space $V = \{e_1, e_2, ..., e_n\}$ satisfying:

$$\{e_i, e_j\} \equiv e_i e_j + e_j e_i = 2\eta_{ij}\mathbf{I} \tag{161}$$

Here the vector space V is called the generating space of $Cl(p,q)$. $\eta_{ij} = 0$ if $i \neq j$, $\eta_{ii} = 1$ for $i = 1, ..., p$, $\eta_{ij} = -1$ for $i = p+1, ..., p+q$. There will be an identity vector in the algebra, such that $a.1 = 1.a = a$, $\forall a \in Cl(p,q)$. The elements of a Clifford algebra can be constructed by multiplication of generating vectors, therefore, it can be be checked that the dimension of the Clifford algebra $Cl(p,q)$ will be $\sum_{i=0}^{n} {}^nC_i = 2^n$.

We give some examples now; the $Cl(0,0)$ algebra will correspond to the real numbers $\mathbf{R}$. There are no generating vectors, just the trivial identity vector in this case. $Cl(0,1)$ will correspond to the complex numbers, here we have only one generating vector with $i^2 = -1$. For $Cl(0,2)$ we need two generating vectors $e_1^2 = -1$ and $e_2^2 = -1$. The algebra will then be $Cl(0,2) = \{1, e_1, e_2, e_1e_2\}$, we identify this algebra with the division algebra of quaternions $\mathbf{H}$. Therefore from the first three Clifford algebras we obtain the division algebras of $\mathbf{R}, \mathbf{C}$, and $\mathbf{H}$. The octonions $\mathbf{O}$ do not make Clifford algebras naturally because of their non-associativity but they can be used to make the associative algebra $Cl(6)$ by introducing the octonionic chains, as is shown by Furey [100].

The next Clifford algebra in this series is $Cl(0,3)$, the generating vectors will be $e_1^2 = -1, e_2^2 = -1$, and $e_3^2 = -1$. The Clifford algebra $Cl(0,3)$ will have eight elements $\{1, e_1, e_2, e_3, e_1e_2, e_2e_3, e_3e_1, e_1e_2e_3\}$. We can break this algebra into sum of two quaternionic parts:

$$(1, e_1, e_2, e_1e_2), \quad (e_1e_2e_3, e_2e_3, e_3e_1, e_3) \tag{162}$$

It is important to understand that $e_3 \neq e_1e_2$ unlike the case for quaternions. We can identify the left set as the quaternions; in the right set $(e_1e_2e_3)^2 = 1$ whereas the square of the other three elements are -1. If we call $e_1e_2e_3$ as ω then the right set becomes $\omega(1, -e_1, -e_2, -e_1e_2)$. Thus the right set is ω times the quaternions. ω here is a split complex number; analogous to i which squares to -1, the split complex numbers square to 1 but are neither 1 nor -1. The conjugate of ω is $-\omega$. Thus, the algebra $Cl(0,3)$ is called the split-biquaternions, as named thus by Clifford himself [99]. It can be written as $\mathbf{D} \otimes \mathbf{H}$, where $\mathbf{D} \equiv (1, \omega)$. Notice that $\mathbf{D} \otimes \mathbf{H} \cong \mathbf{H} \oplus \mathbf{H}$.

Next we consider Clifford algebras on the complex field. The Clifford algebra $Cl(n)$ is defined on the complex field by an n dimensional vector space $V = \{e_1, ..., e_n\}$ such that:

$$\{e_i, e_j\} = 0, \quad i \neq j \tag{163}$$

$$e_i^2 = 1. \tag{164}$$

It is interesting to note that the Clifford algebra $Cl(n)$ on the complex field can be obtained from the Clifford algebra $Cl(p, n-p)$ on the real field by the following relation: $Cl(n) = \mathbf{C} \otimes Cl(p, n-p)$ where $0 \leq p \leq n$.

Therefore, we note that we can get the algebra $Cl(3)$ from complexification of Cl(0,3) and the final algebra would be complex split-biquaternions $\mathbf{C} \otimes (\mathbf{H} \oplus \omega\mathbf{H})$. The Clifford algebra $Cl(0,7)$ is the algebra of split 8×8 real matrices, $\mathbf{R}[8] \oplus \omega\mathbf{R}[8]$. On complexifying we will get $Cl(7)$ which is

$\mathbf{C}[8] \oplus \omega\mathbf{C}[8]$. It has been shown by Furey in [25] that $Cl(6) = \mathbf{C}[8] \cong \overleftarrow{\mathbf{C}} \otimes \overleftarrow{\mathbf{O}}$. Here $\overleftarrow{\mathbf{C}} \otimes \overleftarrow{\mathbf{O}}$ is the algebra of complex octonionic chains, defined as a series of maps acting on a function $f \in \mathbf{C} \otimes \mathbf{O}$ from left to right. Therefore $Cl(7) \cong \overleftarrow{\mathbf{C}} \otimes \overleftarrow{\mathbf{O}} + \omega\overleftarrow{\mathbf{C}} \otimes \overleftarrow{\mathbf{O}}$, the complex split bioctonions.

It is significant that such splitting of the algebra is present only in $Cl(0,3)$ and $Cl(0,7)$. Both of these algebras have two irreducible representations. The irreducible representations of the complex Clifford algebra $Cl(n)$ are called pinors. We note that for $Cl(3)$ and $Cl(7)$ if one of the pinor group is left handed, the other will be right handed, this can be checked from the fact that the split $\omega\mathbf{H}$ is written as $\omega(1, -e_1, -e_2, -e_1e_2)$ and the signs of the imaginary coordinates have flipped. For a more comprehensive analysis of Clifford algebras refer to [41, 101].

A. Complex Quaternions $Cl(2)$ Revisted

The $Cl(2)$ algebra can be obtained by complexifying $Cl(0,2)$ and hence it is identified by complex quaternions $\mathbf{C} \otimes \mathbf{H}$. As we saw, it has been shown by various authors that division algebras can be used to obtain the standard model quarks and leptons [25, 44, 46, 87]. Complex quaternions can give us one generation of leptons, whereas three inequivalent quaternionic subsets of $Cl(6)$ can give us the three generations of leptons [46, 48]. Let us recall the already worked out physics of the $Cl(2)$ algebra.

The generating vector space of $Cl(2)$ algebra is $W = \{ie_1, ie_2\}$. A subspace $U \subset W$ is called a maximal totally isotropic subspace (MTIS) of W, if:

$$\{\alpha_i, \alpha_j\} = 0 \quad \forall \alpha_i \in U \tag{165}$$

We can see that the MTIS for $Cl(2)$ will be one dimensional with either the element $\alpha = (e_1 + ie_2)/2$ or $(-e_1 + ie_2)/2$. The primitive idempotent V of this algebra can be defined as $\alpha\alpha^\dagger$. We can now create minimal left ideals of $Cl(2)$ by left multiplication of $\mathbf{C} \otimes \mathbf{H}$ on the idempotent. The left ideals so obtained will be Weyl spinors of distinct chirality and spin.

If we use $\alpha = (e_1 + ie_2)/2$ to make the idempotent the resulting Weyl spinor will be:

$$\psi_R = \epsilon^{\downarrow\uparrow}\alpha^\dagger V + \epsilon^{\uparrow\uparrow} V \tag{166}$$

where $\epsilon^{\downarrow\uparrow}, \epsilon^{\uparrow\uparrow} \in \mathbf{C}$, V is the idempotent. This right-handed Weyl spinor ψ_R can be interpreted as an entangled state of V and $\alpha^\dagger V$, where V and $\alpha^\dagger V$ are two different leptons. The difference

between these two leptons is evident from the number operator

$$N = \alpha^\dagger \alpha \tag{167}$$

The spinors V and $\alpha^\dagger V$ are eigen-vectors of this Hermitian number operator with eigenvalues 0 and 1 respectively. These eigenvalues correspond to the charge of the leptons. Hence V will be interpreted as the neutrino and $\alpha^\dagger V$ will be interpreted as the charged lepton.

Similarly we can obtain a left-handed Weyl spinor ψ_L if we use the MTIS $\alpha = (-e_1 + ie_2)/2$. The idempotent thus obtained is V^*. Instead of $\alpha^\dagger$; we will now use α to obtain the excited state from the idempotent. The resulting left-handed Weyl spinor is

$$\psi_L = \epsilon^{\uparrow\downarrow}\alpha V^* + \epsilon^{\downarrow\downarrow}V^* \tag{168}$$

where $\epsilon^{\uparrow\downarrow}, \epsilon^{\downarrow\downarrow} \in \mathbf{C}$. Individually we can identify V^* and αV^* as a neutrino and a charged lepton. The left handed leptons will be the anti-particles of right-handed leptons and will be related through complex conjugation $*$.

As we have seen, the SL(2,C) symmetry responsible for Lorentz invariance of vectors and spinors is already present in the Cl(2) algebra. We can write a four vector in Minkowski spacetime using the quaternions as follows:

$$V = v_0 + v_1 e_1 + v_2 e_2 + v_3 e_1 e_2 \tag{169}$$

Any number s in the $\mathbf{C} \otimes \mathbf{H}$ algebra which does not have a component along the real direction can be shown to be the generator of the Lorentz algebra [25]. For $s = s_1 e_1 + s_2 ie_1 + s_3 e_2 + s_4 ie_2 + s_5 e_1 e_2 + s_6 ie_1 e_2$, the Lorentz operator e^{is} will bring about boosts and rotations. We recall that $ie_1, ie_2, ie_1 e_2$ are the Pauli matrices and the subgroup SU(2) is already present in Sl(2,C). This SU(2) is the usual spin group responsible for rotational symmetry in 3-D space.

In their work Gillard and Gresnigt [46] showed that we can get three generations of leptons from three quaternionic subsets of $Cl(6)$. The subtle point to note in their work is that there has to be one common octonion in all three subsets; this is necessary for defining a charge operator for three generations. Manogue and Dray also discussed three generations of leptons from three inequivalent sets of quaternions [48]. We believe that the three generation problem can be understood through rotations of octonions caused by G_2 group and through the exceptional Jordan matrices $J_3(O)$. The triality of $SO(8)$ is the likely explanation as to why there are three fermion generations.

We will now see how we can create one generation of left-right symmetric leptons from $Cl(3)$.

B. The complex Split Biquaternions $Cl(3)$ and L-R symmetric leptons

$Cl(3)$ algebra can be obtained by complexification of $Cl(0,3)$ algebra. We have seen above that $Cl(0,3)$ is the spilt biquaternions, therefore we will call $Cl(3)$ as complex split biquaternions $\mathbf{C}\otimes\mathbf{D}\otimes\mathbf{H}$. Recall that we wrote $Cl(0,3)$ as $\mathbf{D}\otimes\mathbf{H}$, this is isomorphic to $\mathbf{H}\oplus\mathbf{H}$, and the ω is usually left out. However, we will subsequently see the physical importance of ω. It seems that the symmetry laws will be same for $\mathbf{H}$ and $\omega\mathbf{H}$ (which we call as the omega space), because the multiplication in any Lie algebra has two terms and the ω will get squared to one. Despite the fact that the two spaces will have similar symmetry laws, the two spaces are different in terms of chirality and ω plays a crucial role in understanding the constant interaction of left-right fermions with the Higgs.

Above, we wrote the Cl(0,3) algebra as a sum of two sets:

$$(1, e_1, e_2, e_1e_2), \quad \omega(1, -e_1, -e_2, -e_1e_2) \tag{170}$$

It is important to note that the two sets have opposite parity. If we create leptons from the two complex quaternions in $Cl(3)$ we will get two sets of leptons with opposite chirality. The MTIS for the left set of complex quaternions will be either $\alpha = \frac{e_1+ie_2}{2}$ or $\frac{-e_1+ie_2}{2}$. The particles and anti-particles created will be

$$\overline{\mathcal{V}}_R = \frac{1+ie_1e_2}{2} \qquad \mathcal{V}_L = \frac{1-ie_1e_2}{2} \tag{171}$$

$$e_R^+ = \frac{-e_1+ie_2}{2} \qquad e_L^- = \frac{-e_1-ie_2}{2} \tag{172}$$

Hence we have created the left-handed neutrino $\mathcal{V}_L$ and electron e_L^- along with their right-handed anti-particles; the anti-neutrino $\overline{\mathcal{V}}_R$ and positron e_R^+. The particles and anti-particles are related to each other through complex conjugation $*$.

Similarly we can get leptons from the right set of complex quaternions. The MTIS will be either $\alpha = \omega(-e_1 - ie_2)/2$ or $\omega(e_1 - ie_2)/2$. The particles and anti-particles created will be

$$\overline{\mathcal{V}}_L = \frac{1+ie_1e_2}{2} \qquad \mathcal{V}_R = \frac{1-ie_1e_2}{2} \qquad e_L^+ = \omega\frac{(e_1-ie_2)}{2} \qquad e_R^- = \omega\frac{(e_1+ie_2)}{2} \tag{173}$$

As stated before the particles created from the right set of complex quaternions have opposite

chirality to the particles created from left set of complex quaternions. Therefore, we now have the right-handed neutrino $\mathcal{V}_R$ and the right-handed electron e_R^-, along with their left-handed anti-particles; the anti-neutrino $\overline{\mathcal{V}}_L$ and positron e_L^+. It has been pointed in [25] that parity transformation can be brought by $e_i \to -e_i$, our result here is consistent with this fact. The generator for $U(1)_{em}$

$$Q = \alpha^\dagger \alpha \tag{174}$$

is present in $Cl(3)$ and provides charge to the leptons of both the sectors.

A left-right symmetric model for particle physics has been long proposed by many authors [80, 84]. The electron that we detect in our experiments is in constant interaction with the Higgs boson. All the leptons created from the left set of quaternions are either left-handed particles or their right-handed anti-particles, therefore they have a weak hypercharge and interact with the weak bosons through $SU(2)_L$ symmetry. The leptons created from the right set of complex quaternions on the other hand cannot interact with the weak bosons. The Higgs acts as a source and sink for hypercharge and changes the right-handed electron to left-handed, and the left-handed electron to right-handed.

It is significant that the ω in $Cl(3)$ maps the left-handed electron to the right-handed electron and conversely the right-handed electron to the left-handed electron. It is also worth noting that ω is self-adjoint

$$(e_1 e_2 e_3)^\dagger = e_1 e_2 e_3 \tag{175}$$

Therefore, there is a $U(1)$ symmetry associated with ω, similar to the $U(1)$ symmetry associated with the Higgs boson in the standard model. It could well be that the split complex number ω is related to the Higgs boson.

Right-handed electrons and quarks are known in the current standard model but the right-handed neutrinos remain to be a mystery. It seems that all the neutrinos detected till now are left-handed in nature. It has been proposed that a right-handed neutrino should explain the mass of neutrinos through a see-saw mechanism [79]. The see-saw mechanism predicts that the right-handed neutrino should be significantly heavier than the left-handed neutrino. Since these right-handed neutrinos do not interact through any other force apart from gravity they are called sterile neutrinos and are a potential dark matter candidate. Experiments such as the neutrinoless double

beta decay have been proposed to infer the possible existence of right-handed sterile neutrinos [79, 80]. It is also interesting to note that the neutrino is required to be a Majorana neutrino for explaining the see-saw mechanism. Our work on mass ratios of fermions also requires the neutrino to be Majorana [72].

It has also been proposed that the right-handed particles should interact with the right analogues of weak bosons through $SU(2)_R$ symmetry, but these bosons manifest themselves only at very high energy scales. The corresponding gauge group for left-right symmetric fermions $SU(3)_c \times SU(2)_R \times SU(2)_L \times U(1)_{em}$ is given by the Pati-Salam model [84]. We will see subsequently how this emerges from right action of $\mathbf{C} \otimes \mathbf{H}$ on $Cl(7)$.

There is a difference however between the Pati-Salam model and the model that we will propose through division algebras. Pati-Salam model is a grand unified theory (GUT) on a Minkowski space-time background. We insist that unlike in quantum field theory the division algebra approach to standard model does not require a background spacetime. In spite of this fact the Lorentz symmetry is already present in $\mathbf{C} \otimes \mathbf{H}$, as has been shown in section 2. We also do not impose any quantum condition in an ad-hoc manner, the quantization of charge comes naturally from the number operator. Therefore, physics through division algebras can be a gateway to pre-spacetime, pre-quantum theories with unification of all gauge fields including gravity [6, 50, 72, 102].

We conclude that the Clifford algebra $Cl(3)$ can give us one generation of left-right symmetric leptons with $U(1)$ symmetry for photon and Higgs boson. In the next subsection we study the $Cl(7)$ algebra and describe one generation of fermions with symmetry $SU(3) \times U(1)$.

C. The Complex Split Bioctonions $Cl(7)$

As we saw, the relation between quarks and division algebras has intrigued several authors in the past [23, 25–27, 44–46, 87]. Furey introduced the octonionic chain algebra to relate the non-associative octonions with $Cl(6)$. We have already defined the octonionic chains $\overleftarrow{\mathbf{C}} \otimes \overleftarrow{\mathbf{O}}$ above. Throughout our analysis we will take the function $f \in \mathbf{C} \otimes \mathbf{O}$ to be 1, so our octonionic chains act on 1.

The generators for $Cl(6)$ will be $\{i\overleftarrow{e_1}, i\overleftarrow{e_2}, i\overleftarrow{e_3}, i\overleftarrow{e_4}, i\overleftarrow{e_5}, i\overleftarrow{e_6}\}$. We are dropping the left-arrow over octonions now, but throughout our analysis we will be working with octonionic chains. The MTIS will be 3-dimensional with the following elements:

$$\alpha_1 = \frac{-e_5 + ie_4}{2}, \quad \alpha_2 = \frac{-e_3 + ie_1}{2}, \quad \alpha_3 = \frac{-e_6 + ie_2}{2} \tag{176}$$

If we define $\Omega = \alpha_1\alpha_2\alpha_3$, then the idempotent will be $\Omega\Omega^\dagger = \alpha_1\alpha_2\alpha_3\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger$. On left-multiplying the idempotent with elements of the MTIS we obtain these excited states:

$$\overline{\mathcal{V}} = \Omega\Omega^\dagger = \frac{ie_7+1}{2} \tag{177}$$

$$V_{ad1} = \alpha_1^\dagger\mathcal{V} = \frac{e_5+ie_4}{2} \tag{178}$$

$$V_{ad2} = \alpha_2^\dagger\mathcal{V} = \frac{e_3+ie_1}{2} \tag{179}$$

$$V_{ad3} = \alpha_3^\dagger\mathcal{V} = \frac{e_6+ie_2}{2} \tag{180}$$

$$V_{u1} = \alpha_3^\dagger\alpha_2^\dagger\mathcal{V} = \frac{e_4+ie_5}{2} \tag{181}$$

$$V_{u2} = \alpha_1^\dagger\alpha_3^\dagger\mathcal{V} = \frac{e_1+ie_3}{2} \tag{182}$$

$$V_{u3} = \alpha_2^\dagger\alpha_1^\dagger\mathcal{V} = \frac{e_2+ie_6}{2} \tag{183}$$

$$V_{e+} = \alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger\mathcal{V} = -\frac{(i+e_7)}{2} \tag{184}$$

The following generator for $U(1)_{em}$ provides charge to the quarks and leptons

$$Q = \frac{\alpha_1^\dagger\alpha_1 + \alpha_2^\dagger\alpha_2 + \alpha_3^\dagger\alpha_3}{3} \tag{185}$$

Therefore we get one generation of quarks and leptons from $Cl(6)$; the anti-particles will be related through complex conjugation $*$. As can be seen, each quark comes in three colours. It has been shown by authors [25–27, 46] that the algebra $\overleftarrow{\mathbf{C}} \otimes \overleftarrow{\mathbf{O}}$ already has the symmetry group $SU(3)$ present in it. The $SU(3)$ generators in terms of octonions are given by:

$$\Lambda_1 = -\alpha_2^\dagger\alpha_1 - \alpha_1^\dagger\alpha_2 \qquad \Lambda_5 = -i\alpha_1^\dagger\alpha_3 + i\alpha_3^\dagger\alpha_1 \tag{186}$$

$$\Lambda_2 = i\alpha_2^\dagger\alpha_1 - i\alpha_1^\dagger\alpha_2 \qquad \Lambda_6 = \alpha_3^\dagger\alpha_2 - \alpha_2^\dagger\alpha_3 \tag{187}$$

$$\Lambda_3 = \alpha_2^\dagger\alpha_2 - \alpha_1^\dagger\alpha_1 \qquad \Lambda_7 = i\alpha_3^\dagger\alpha_2 - i\alpha_2^\dagger\alpha_3 \tag{188}$$

$$\Lambda_4 = -\alpha_1^\dagger\alpha_3 - \alpha_3^\dagger\alpha_1 \qquad \Lambda_8 = -\frac{(\alpha_1^\dagger\alpha_1 + \alpha_2^\dagger\alpha_2 - 2\alpha_3^\dagger\alpha_3)}{\sqrt{(3)}} \tag{189}$$

It can be checked using the charge generator that Ω has a charge -1, whereas $\Omega^\dagger$ has a charge +1. Right multiplication of Ω on the particles (eq. 21-28) changes their isospin from up to down, whereas the right multiplication of $\Omega^\dagger$ on the anti-particles changes their isospin from down to up. Therefore, Ω mimics the W^- boson and $\Omega^\dagger$ mimics the W^+ boson.

It is significant that using $Cl(6)$ we can create one generation of fermions with distinct chiral-

ity. Left-handed particles from $Cl(6)$ will have right-handed anti-particles. We will now investigate $Cl(7)$ to see that it can be naturally written as a sum of two $\overleftarrow{\mathbf{C}} \otimes \overleftarrow{\mathbf{O}}$ with opposite parity. Throughout our analysis octonions should be treated as octonionic chains acting on the function $f = 1$; it is important to note that unlike in $Cl(6)$ $\overleftarrow{e_7} \neq \overleftarrow{e_1e_2e_3e_4e_5e_6}$. For our convenience we replace $\overleftarrow{e_1e_2e_3e_4e_5e_6}$ by $\overleftarrow{e_8}$. Cl(0,7) can be made from two sets of octonions:

$$(1, e_1, e_2, e_3, e_4, e_5, e_6, e_8) \oplus \omega(1, -e_1, -e_2, -e_3, -e_4, -e_5, -e_6, -e_8) \tag{190}$$

Here $\omega = \overleftarrow{e_1e_2e_3e_4e_5e_6e_7}$. The above line of reasoning is not direct to see, but we can understand it in the following way. $Cl(0,7)$ is split 8×8 real matrices $R[8] \oplus R[8]$, therefore $Cl(7)$ will be $C[8] \oplus C[8]$, the Clifford algebra $Cl(6)$ is also $C[8]$ and is isomorphic to complex octonionic chains. Therefore $Cl(7)$ will be isomorphic to complex split octonionic chains. Just like $e_1e_2e_3$ commutes with all elements in $Cl(3)$, $\overleftarrow{e_1e_2e_3e_4e_5e_6e_7}$ will commute with all the elements of $Cl(7)$, and it squares to 1.

We can now make particles from the left set and the right set in a similar way as is done in $Cl(6)$. The MTIS for the left set will be $\alpha_1 = \frac{-e_5+ie_4}{2}, \alpha_2 = \frac{-e_3+ie_1}{2}, \alpha_3 = \frac{-e_6+ie_2}{2}$. The idempotent will be $\Omega_L\Omega_L^\dagger = \alpha_1\alpha_2\alpha_3\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger$. The left-handed neutrino family and their right-handed anti-particles are:

$$\overline{\mathcal{V}} = \frac{ie_8+1}{2} \qquad \mathcal{V} = \frac{-ie_8+1}{2} \tag{191}$$
$$V_{ad1} = \frac{(e_5+ie_4)}{2} \qquad V_{d1} = \frac{(e_5-ie_4)}{2} \tag{192}$$
$$V_{ad2} = \frac{(e_3+ie_1)}{2} \qquad V_{d2} = \frac{(e_3-ie_1)}{2} \tag{193}$$
$$V_{ad3} = \frac{(e_6+ie_2)}{2} \qquad V_{d3} = \frac{(e_6-ie_2)}{2} \tag{194}$$
$$V_{u1} = \frac{(e_4+ie_5)}{2} \qquad V_{au1} = \frac{(e_4-ie_5)}{2} \tag{195}$$
$$V_{u2} = \frac{(e_1+ie_3)}{2} \qquad V_{au2} = \frac{(e_1-ie_3)}{2} \tag{196}$$
$$V_{u3} = \frac{(e_2+ie_6)}{2} \qquad V_{au3} = \frac{(e_2-ie_6)}{2} \tag{197}$$
$$V_{e+} = -\frac{(i+e_8)}{2} \qquad V_{e-} = -\frac{(-i+e_8)}{2} \tag{198}$$

The MTIS for the right set will be $\alpha_1 = -\omega\frac{-e_5+ie_4}{2}$, $\alpha_2 = -\omega\frac{-e_3+ie_1}{2}$, $\alpha_3 = -\omega\frac{-e_6+ie_2}{2}$. The idempotent will be $\Omega_R\Omega_R^\dagger = \alpha_1\alpha_2\alpha_3\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger$. The right-handed neutrino family and their left-

handed anti particles are:

$$\overline{\mathcal{V}} = \frac{ie_8+1}{2} \qquad \mathcal{V} = \frac{ie_8+1}{2} \tag{199}$$

$$V_{ad1} = \omega\frac{(-e_5-ie_4)}{2} \qquad V_{d1} = \omega\frac{(-e_5+ie_4)}{2} \tag{200}$$

$$V_{ad2} = \omega\frac{(-e_3-ie_1)}{2} \qquad V_{d2} = \omega\frac{(-e_3+ie_1)}{2} \tag{201}$$

$$V_{ad3} = \omega\frac{(-e_6-ie_2)}{2} \qquad V_{d3} = \omega\frac{(-e_6+ie_2)}{2} \tag{202}$$

$$V_{u1} = \frac{(e_4+ie_5)}{2} \qquad V_{au1} = \frac{(e_4-ie_5)}{2} \tag{203}$$

$$V_{u2} = \frac{(e_1+ie_3)}{2} \qquad V_{au2} = \frac{(e_1-ie_3)}{2} \tag{204}$$

$$V_{u3} = \frac{(e_2+ie_6)}{2} \qquad V_{au3} = \frac{(e_2-ie_6)}{2} \tag{205}$$

$$V_{e+} = \omega\frac{(i+e_8)}{2} \qquad V_{e-} = \omega\frac{(-i+e_8)}{2} \tag{206}$$

Therefore from the two sets on $Cl(7)$ we have one generation of left-right symmetric fermions. Both the right sector and the left sector have $SU(3) \times U(1)$ symmetry.

Just as mentioned before, Ω_L and $\Omega_L^\dagger$ will play the role of W^- and W^+ bosons respectively. On the contrary, Ω_R and $\Omega_R^\dagger$ will play the role of right-handed analogues of weak bosons. In the next section we see why $SU(2)_L$ acts only on left-handed particles and their right-handed anti-particles whereas $SU(2)_R$ acts only on right-handed particles and their left-handed anti-particles.

It is noteworthy that ω in $Cl(7)$ can explain the interaction of left-right symmetric fermions with the Higgs similar to the ω in $Cl(3)$ but with an addition of quarks. In case of up quarks however the ω gets squared to 1. The absence of ω for neutrinos and up quarks, and the fact that the ratio of square root of mass of electron, up-quark, and down quark is $\frac{1}{3} : \frac{2}{3} : 1$ motivated the authors to talk about mass ratios for one generation of lepto-quarks prior to symmetry breaking, in section 6.

D. Why do the $SU(2)_L$ and $SU(2)_R$ act only on left-handed or right-handed fermions?

Our universe seems to show some bias towards the left-handed fermions in the sense that weak bosons interact only with left-handed fermions. In quantum field theory this is accepted in an ad-hoc manner on experimental grounds. Furey in her work [25–27] showed why only left-handed fermions interact through $SU(2)_L$. In Pati-Salam model we have both $SU(2)_L$ and $SU(2)_R$, in this section we extend Furey's work to show why only right-handed fermions interact through $SU(2)_R$

and left-handed fermions interact through $SU(2)_L$. We have discussed in previous subsections that the right action of Ω and $\Omega^\dagger$ reverses the isospin of particles. The right action of $\mathbf{C}\otimes\mathbf{H}$ on $Cl(3)$ and $Cl(7)$ changes the chirality and isospin of the particles [25, 26]. In doing so we map to the Clifford algebras $Cl(5)$ and $Cl(9)$ respectively. The right action of $\mathbf{C}\otimes\mathbf{H}$ on $\mathbf{C}\otimes\mathbf{H}$ will look like

$$(1, e_1, e_2, e_1e_2) \otimes_C (1, e_1, e_2, e_1e_2) \tag{207}$$

This will give us the $Cl(4)$ algebra, which we can generate using the vectors

$$\{\tau_1 ie_2, \tau_2 ie_2, \tau_3 ie_2, ie_1\} \tag{208}$$

Here $\tau_1 = \Omega_L + \Omega_L^\dagger$, $\tau_2 = i\Omega_L - i\Omega_L^\dagger$, $\tau_3 = \Omega_L\Omega_L^\dagger - \Omega_L^\dagger\Omega_L$. Here Ω_L is borrowed from our previous section. We note that the MTIS in this case will be two dimensional spanned by

$$\beta_1 = \frac{-e_1 + ie_2\tau_3}{2}, \quad \beta_2 = \Omega_L^\dagger ie_2 \tag{209}$$

The idempotent will be $\beta_1^\dagger\beta_2^\dagger\beta_2\beta_1 = \mathcal{V}_R$. We can obtain the right ideals by right multiplication of $Cl(4)$ on this ideal. We identify the following particles

$$\beta_1^\dagger\beta_2^\dagger\beta_2\beta_1 = \mathcal{V}_R; \quad e_R^- = \mathcal{V}_R\beta_1^\dagger\beta_2^\dagger \tag{210}$$

$$e_L^- = \mathcal{V}_R\beta_2^\dagger; \quad \mathcal{V}_L = \mathcal{V}_R\beta_1^\dagger \tag{211}$$

The SU(2) symmetry can be generated by the following three generators

$$T_1 = \tau_1\frac{1+ie_1e_2}{2}, T_2 = \tau_2\frac{1+ie_1e_2}{2}, T_3 = \tau_3\frac{1+ie_1e_2}{2} \tag{212}$$

It is interesting to note that the SU(2) operators will annihilate the particles $\mathcal{V}_R$ and e_R^- (eq. 54), whereas it will interchange e_L^- and $\mathcal{V}_L$ as isospin states. Therefore, $SU(2)_L$ acts on left-handed particles or conversely on right-handed anti-particles.

Similarly we can understand the right action of $\mathbf{C}\otimes\mathbf{H}$ on $\omega\mathbf{C}\otimes\mathbf{H}$. This can be written as

$$(1, -e_1, -e_2, -e_1e_2) \otimes_C (1, e_1, e_2, e_1e_2) \tag{213}$$

The algebra will again be $Cl(4)$, the particles generated through similar analysis will be

$$\beta_1^\dagger \beta_2^\dagger \beta_2 \beta_1 = \mathcal{V}_L; \quad e_L^- = \mathcal{V}_L \beta_1^\dagger \beta_2^\dagger \tag{214}$$

$$e_R^- = \mathcal{V}_L \beta_2^\dagger; \quad \mathcal{V}_R = \mathcal{V}_L \beta_1^\dagger \tag{215}$$

The $SU(2)$ generators will now annihilate $\mathcal{V}_L$ and e_L^- (eq. 54), whereas it will interchange e_R^- and $\mathcal{V}_R$ as isospin states. Therefore, $SU(2)_R$ acts on right-handed particles or conversely on left-handed anti-particles. Up until now we have discussed one generation of left-right symmetric fermions. In the next subsection we discuss three generations of left-right symmetric fermions and the corresponding gauge-groups.

E. Inclusion of Gravity through division algebras

The automorphism group of the octonions is G_2, it has a total of 14 generators. Eight of these are the generators of $SU(3)$ given by equation (30-33). The remaining six generators are given by:

$$g_1 = -\frac{i}{2\sqrt{3}}(e_1(e_5.) + e_3(e_4.) + 2e_2(e_7.)) \tag{216}$$

$$g_2 = \frac{i}{2\sqrt{3}}(e_1(e_4.) - e_3(e_5.) + 2e_6(e_7.)) \tag{217}$$

$$g_3 = -\frac{i}{2\sqrt{3}}(e_4(e_6.) - e_2(e_5.) + 2e_1(e_7.)) \tag{218}$$

$$g_4 = -\frac{i}{2\sqrt{3}}(e_2(e_4.) + e_5(e_6.) - 2e_3(e_7.)) \tag{219}$$

$$g_5 = -\frac{i}{2\sqrt{3}}(-e_1(e_6.) + e_2(e_3.) + 2e_4(e_7.)) \tag{220}$$

$$g_6 = \frac{i}{2\sqrt{3}}(-e_1(e_2.) + e_3(e_6.) + 2e_5(e_7.)) \tag{221}$$

These values have been taken from [25], the brackets indicate that the multiplication is from left to right and the octonions are actually octonionic chains. These six generators form another maximal subgroup of G_2, namely $SO(4)$. The $SO(4)$ group is isomorphic to $SU(2) \times SU(2)$ and hence has six generators. In [50], we discussed how this group structure of G_2 can lie at the heart of Lorentz-weak unification. The group $SU(3) \times U(1)$ gives us electro-colour symmetry whereas $SO(4)$ gives us the Lorentz-weak symmetry. Stoica talks about two types of $Cl(6)$ algebra for weak symmetry and electro-colour symmetry in [45], Furey uses $Cl(6)$ to show the $SU(3)\times U(1)$ maximal subgroup of G_2. We propose that prior to electroweak symmetry breaking the $SU(2)_R$ allows us to build (right-handed) Lorentz symmetry whereas $SU(2)_L$ gives us weak symmetry, therefore the group

$SO(4)$ gives us Lorentz-weak unification.

Just as we are associating photon with the $U(1)$ group, gluons with the $SU(3)$ group, and weak bosons with the $SU(2)_L$ group, we propose associating the $SU(2)_R$ group with two spin one Lorentz bosons for gravity. These Lorentz bosons are on an equal footing with the other 12 gauge bosons, and these bosons manifest themselves in a pre-spacetime, pre-quantum theory called generalized trace dynamics [50, 103]. As noted in section 6 there are fourteen generators of G_2, therefore getting 14 spin one gauge bosons seems very promising. Another interesting thing to note is the counting of fermionic and bosonic degrees of freedom.

Left-right symmetric fermions (dof): $[8 + 8] \times 2 = 32$, the factor of 2 is for the spin.

Bosons (dof): 8 gluons = $8\times 2 = 16$, 3 massive weak bosons = $3\times 3 = 9$, photon = 2, and Higgs boson = 1. Therefore we do not have a total of 32 bosonic degrees of freedom, as this adds up only to 28, falling four short of the 32 fermionic dof. The bosonic dof become exactly 32 after we include 2x2=4 for the two massless spin one Lorentz bosons. The match between bosonic and fermionic degrees of freedom is in accordance with generalized trace dynamics.

Prior to symmetry breaking, space-time is not 4D, but an eight dimensional octonionic space-time [equivalently 10D Minkowski spacetime] labelled by the octonions. There is a unification of the Lorentz-weak symmetry with the electro-color symmetry and the proposed symmetry group is E_6. However, we do not have a GUTs. We have an L-R symmetric pre-quantum pre-spacetime theory in which the internal symmetries and the Lorentz symmetry of 4D spacetime have been unified via an 8D octonionic Kaluza-Klein theory. The dynamics is generalised trace dynamics, from which quantum field theory is emergent after symmetry breaking. In the unified theory, the concept of electric charge and square-root mass merge into one: charge-root-mass, which is quantised in units of 0, 1/3, 2/3 and 1, and comes with both signs, the negative sign being for anti-particles.

Lepto-quarks are bosons. We can possibly account for the 78 dimensions of E_6 as follows. (8+8)x3 = 48 lepto-quarks plus 14 gauge bosons adds up to 62. The remaining 16 dimensions could possibly be assigned to the octonionic projective plane OP^2 occupied by the lepto-quarks. This would be in keeping with treating matter, gauge and space-time aspects all on the same footing ['atoms' of space-time-matter]. The lepto-quarks will be described by the Clifford algebra $Cl(7)$ related to two copies of $Cl(6)$ and the split bioctonions. This underscores the importance of the split bioctonions studied in the present paper. Whereas in earlier sections the two copies of $Cl(6)$ both deal with electric charge as the quantum number, in the pre-theory one copy deals with electric charge, and the other with square-root mass, thus bringing gravitation within the framework of division algebras and Clifford algebras.

We would like to comment on the pre-spacetime and pre-quantum nature of division algebras. In the division algebra approach to particle physics we do not impose quantum conditions in an ad-hoc manner, the quantization of charge comes naturally from the number operators. Division algebra approach is also pre-spacetime in nature because we are not assuming any quantum fields defined at each point of spacetime to explain the standard model. In this regard there is a similarity between division algebras and generalized trace dynamics. However division algebras give us only the particles and their underlying symmetry groups, to get into the dynamics of these particles we need to define a Lagrangian as has been done in generalized trace dynamics [50].

XII. E_6 AS A POSSIBLE SYMMETRY GROUP FOR THREE GENERATIONS, AND WOULD-BE-GRAVITY AS THE RIGHT HANDED COUNTERPART OF THE STANDARD MODEL

The discussion in this section is motivated by the question: why is the square-root mass ratio 3: 2: 1 of the down quark, up quark and electron in the reverse order of their electric charge ratio 1: 2: 3? We believe that rather than being a coincidence, this fact points to deep physics, and that the symmetry group E_6 has an answer. We will assume that space-time is an eight dimensional manifold labeled by the octonions, and by virtue of the isomorphism $SL(2,\mathbb{O}) \sim SO(9,1)$ this is equivalent to a 10D Minkowski space-time manifold. Three generations of fermions reside on this space-time on which E_6 acting as the symmetry group is a candidate for the unification of the standard model with gravity, as we now argue.

E_6 is the only exceptional Lie group which has complex representations, and it has two maximal subgroups $\tilde{H_1} = [SU(3) \times SU(3) \times SU(3)]/\mathbf{Z_3}$, $\tilde{H_2} = Spin(10)$. Their intersection is $SU(3) \times SU(2)_R \times SU(2)_L \times U(1)$ which is the gauge-group for left-right symmetric model. The groups belonging to the two maximal sub-groups but lying outside the intersection are $Spin(6)$ and $SU(3) \times SU(3)$. We identify one of these two $SU(3)$ with generational symmetry, and now the novel part is that we introduce gravi-color, analogous to QCD color, and associate this third $SU(3)$ with gravitation and square-root mass number. This will help understand the down : up : electron square-root mass ratio of 3: 2: 1 Just as $SU(3)_c \times U(1)_{em}$ is described by the Clifford algebra $Cl(6)$ as unbroken electro-color, the group $SU(3)_{grav} \times U(1)_g$ will describe unbroken gravi-color through another copy of $Cl(6)$ and together these two copies of $Cl(6)$ will form a $Cl(7)$ using the complex split bioctonions [104]. This offers a unification of QCD color with gravi-color, prior to the L-R symmetry breaking, which we assume is the same as the electro-weak symmetry breaking.

The group $SU(2)_L \times SU(2)_R$ describes gravi-weak unification through complex split biquaternions; $SU(2)_L$ is the standard model weak symmetry and $SU(2)_R$ is the gravi- part of gravi-weak, mediated by two gravitationally charged 'Lorentz' bosons, a neutral Lorentz boson, and the Higgs. In our theory there are no right-handed weak bosons; these are replaced by three right-handed Lorentz bosons, and the electro-weak symmetry breaking also breaks the gravi-weak symmetry. The $Spin(6)$ which is not in the intersection is identified as a six dimensional Minkowski space-time because of the isomorphism $Spin(6) \sim SO(5,1) \sim SL(2,H)$. This possibly is the space-time spanned by the gravi-weak interaction.

Prior to L-R symmetry breaking, the neutrino is a Dirac neutrino, which after symmetry breaking separates into the left-handed active Majorana neutrino, and the right-handed sterile Majorana neutrino. Analogous to how it was done in [104], we use the Dirac neutrino as an idempotent, prior to L-R symmetry breaking, and construct the Clifford algebra $Cl(7) = Cl(6) + Cl(6)$ displayed below.

$$\overline{\mathcal{V}}_L = \frac{ie_8 + 1}{2} \qquad \overline{\mathcal{V}}_R = \frac{ie_8 + 1}{2} \tag{222}$$

$$V_{ad1} = \frac{(e_5 + ie_4)}{2} \qquad V_{e+1} = \omega\frac{(-e_5 - ie_4)}{2} \tag{223}$$

$$V_{ad2} = \frac{(e_3 + ie_1)}{2} \qquad V_{e+2} = \omega\frac{(-e_3 - ie_1)}{2} \tag{224}$$

$$V_{ad3} = \frac{(e_6 + ie_2)}{2} \qquad V_{e+3} = \omega\frac{(-e_6 - ie_2)}{2} \tag{225}$$

$$V_{u1} = \frac{(e_4 + ie_5)}{2} \qquad V_{au1} = \frac{(e_4 + ie_5)}{2} \tag{226}$$

$$V_{u2} = \frac{(e_1 + ie_3)}{2} \qquad V_{au2} = \frac{(e_1 + ie_3)}{2} \tag{227}$$

$$V_{u3} = \frac{(e_2 + ie_6)}{2} \qquad V_{au3} = \frac{(e_2 + ie_6)}{2} \tag{228}$$

$$V_{e+} = -\frac{(i + e_8)}{2} \qquad V_{ad} = \omega\frac{(i + e_8)}{2} \tag{229}$$

Notation is as in [104]. The eight fermions on the left are made by using the left-handed anti-neutrino as the idempotent, while the eight fermions on the right are made by using the right-handed anti-neutrino as idempotent. The two sets share a common number $U(1)_{electro-gravi}$ operator defined as usual by

$$Q_{gem} = \frac{\alpha_1^\dagger \alpha_1 + \alpha_2^\dagger \alpha_2 + \alpha_3^\dagger \alpha_3}{3} \tag{230}$$

and have an $SU(3)_c \times SU(3)_{grav}$ symmetry, which we interpret as the unification of QCD color

and gravity, and also of electromagnetism and a $U(1)_{grav}$. Here, Q_{gem} is the gravi-electric-charge number operator: after the symmetry breaking this will be interpreted as the electric charge for the left-handed particles, and square-root mass number for the right handed particles. The $U(1)_{electro-gravi}$ boson will separate into the photon for electromagnetism, and a newly proposed gravitational boson. Prior to symmetry breaking the particle content for one generation is as follows. Anti-particles are obtained by ordinary complex conjugation of the particles, as before.

The Dirac neutrino is the sum of the left handed neutrino and the right handed neutrino; it has $Q_{gem} = 0$, is a singlet under $SU(3)_c \times SU(3)_{grav}$ and we can denote it as the particle LeftHandedNeutrino-RightHandedNeutrino, and after the L-R symmetry breaking it acquires mass and separates into a left-handed active Majorana neutrino and a right handed sterile Majorana neutrino.

The first excitation above the idempotent has $Q_{gem} = 1/3$ and is an anti-triplet under $SU(3)_c$ and an anti-triplet under $SU(3)_{grav}$. We denote this particle as LeftHandedAntiDownQuark-RightHandedPositron. After the L-R symmetry breaking it separates into the left-handed anti-down quark of electric charge 1/3 and right-handed positron of square-root mass number 1/3

The second excitation above the idempotent has $Q_{gem} = 2/3$ and is a triplet under $SU(3)_c$ and a triplet under $SU(3)_{grav}$. We denote this particle as LeftHandedUpQuark-RightHandedUpQuark. After the L-R symmetry breaking it separates into the left-handed up quark of electric charge 2/3 and right-handed up quark of square-root mass number 2/3.

The third excitation above the idempotent has $Q_{gem} = 1$ and is a singlet under both $SU(3)_c$ and $SU(3)_{grav}$. We denote this particle as LeftHandedPositron-RightHandedAntiDownQuark. After the L-R symmetry breaking it separates into a left-handed positron of electric charge 1 and a right-handed anti-down quark of square-root mass number 1.

The corresponding anti-particles have a Q_{gem} number of the opposite sign.

We propose to identify the right-handed positron of square-root mass number 1/3 with the left-handed positron of electric charge 1 as being the same particle. This is essentially a proposal for a gauge-gravity duality which we hope to justify from the dynamics. Similarly, the right-handed anti-down quark with square-root mass number 1 is identified with the left-handed anti-down quark of electric charge 1/3. The right-handed up quark of square-root mass number 2/3 is identified with the left-handed up quark of electric charge 2/3. In this way we recover one generation of standard model fermions after the L-R symmetry breaking.

Before symmetry breaking, we can define $\ln \alpha_{unif} \propto 2 \ln Q_{gem} \equiv \ln(\alpha\beta) = \ln \alpha + \ln \beta \propto q + \sqrt{m}$ where $\ln \alpha$ is proportional to electric charge and $\ln \beta$ is proportional to square-root mass, and at

the time of L-R symmetry breaking $2 \ln Q_{gem}$ separates into two equal parts, one identified with electric charge, and the other with square-root mass. We hence see that in the unified L-R phase we can define a new entity, a charge-root-mass as $\alpha_{unif} = \exp q \exp \sqrt{m} \equiv E\sqrt{M}$. This is the source of the unified force described by a $U(1)$ boson, sixteen gravi-gluons, and six gravi-weak bosons corresponding to $SU(2)_L \times SU(2)_R$ and the Higgs; adding to a total of 24 bosons. There are 48 fermions for three generations, giving a total of 48+24 =72, to which if we add six d.o.f. for the six dimensional space-time $SO(5,1)$ we might be able to account for the 78 dimensional E_6. The gravi-weak bosons generate the Lorentz-weak symmetry by their right action on the $Cl(7)$, as described in [104]. After symmetry breaking this separates into the short range weak interaction and long-range gravity described by general relativity. $SU(3)_{grav}$ is negligible in strength compared to QCD color but plays a very important role of describing the square-root mass number as source of would-be-gravity and showing that mass-quantisation arises only after the standard model has been unified with gravity, as was always anticipated. We also see via E_6 that $SU(3)_{grav} \times SU(2)_R \times U(1)_g$ is the gravitational counterpart of the standard model $SU(3)_c \times SU(2)_L \times U(1)_{em}$. The remaining entities from the two maximal sub-groups, i.e. $SU(3)_{gen}$ and $Spin(6)$ respectively give rise to three generations and a 6D Minkowski space-time. We now finally understand why the square-root mass ratios 3:2:1 for down : up : electron are in the reverse order as the ratio $1:2:3$ of their electric charge. It is a consequence of the gauge-gravity duality afforded by E_6. This duality might throw some light on the color-kinematics relations between QCD and general relativity [105], the results on gravity as square of a Yang-Mills theory [106], and perhaps might also help understand the AdS/CFT correspondence. We also see that fundamentally gravitation is completely different from general relativity. Gravitation is the right-handed counterpart of the standard model, with GR as its low-energy limit. The Higgs mediates between the left-handed electrically charged particles and the right-handed massive particles, transporting charge from left to right, and mass from right to left.

The generational symmetry invoked by $SU(3)_{gen}$ gives rise to three copies of $Cl(7)$ and to the mass ratios derived earlier in this paper. The three generations are distinguished and labeled by their Jordan eigenvalues. The main aspect which now remains to be understood is the right action of the gravi-weak symmetry $SU(2)_L \times SU(2)_R$ on each of the three generations and whether this action maps one generation to another in such a way as to correctly reproduce the known physics of the standard model. This is currently under investigation and sketched in the schematic below in Fig. (13).

Space-time is 8D octonionic equivalent to 10D Minkowski, and this manifold evolves in the

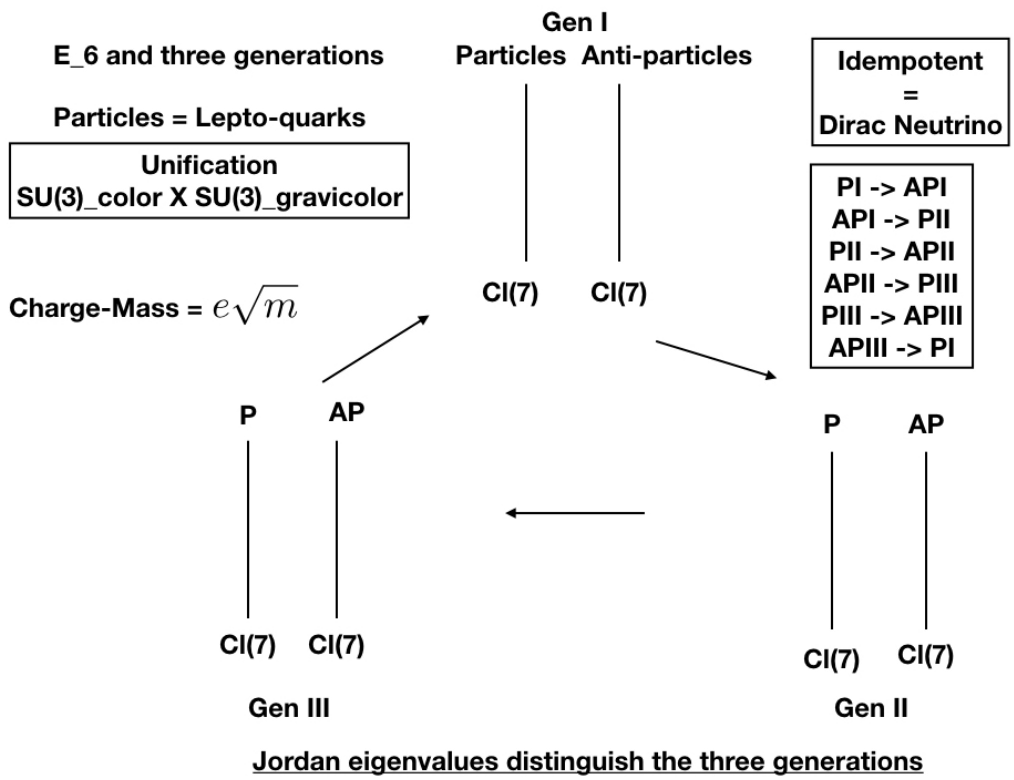


FIG. 13. E_6, three fermion generations with L-R symmetry, and a proposal for unification of the standard model with gravitation. The left-handed sector is the standard model; the right-handed sector is would be gravity. This is a very explicit example of gauge-gravity duality: gauge and gravity get unified before the L-R symmetry breaking [70].

so-called Connes time τ, via the so-called generalised trace dynamics operating as a pre-quantum, pre-spacetime theory before the L-R symmetry breaking [72]. After symmetry breaking the dynamics is conventional quantum field theory, and classical 4D Minkowski spacetime also emerges. The extra dimensions are not compactified because only classical systems live in 4D. Quantum systems continue to live in 10D Minkowski even in today's universe, and the thickness of the extra dimensions is of the order of the support of the wave-function of the said quantum system. If a quantum system tries to become too large, it collapses to 4D and becomes classical. The quantum-to-classical transition is also a transition from ten to four dimensions, and a classical system penetrates the extra dimensions to a depth less than Planck length - which is why it is classical in the first place.

We have earlier [72] constructed a matrix valued Lagrangian in the generalised trace dynamics for describing three fermion generations, with the symmetry group being E_6. This Lagrangian has two coupling constants, a length parameter L which can be related to square-root mass, and a

coupling constant α which can now be identified with the α_{unif} above, prior to the L-R symmetry breaking. After the symmetry breaking, this coupling will become a sum of two parts: one being the coupling constant for electro-color, and the other for $U(1)_g\times$ gravicolor. The gravicolor part of the latter is utterly negligible on small scales in today's universe, where $U(1)_g$ is a newly predicted gravi-boson.

We believe we have put forth a candidate theory for unification worth investigating further, also because E_6 has promising phenomenology [90]. There is no strong CP problem in this theory, and there is also a potential for addressing the origin of matter-antimatter asymmetry. The standard model parameters could get determined by the exceptional Jordan elgebra, with its symmetry group F_4 being in a sense the real part of E_6, and hence using F_4 is like finding expectation values in quantum theory.

A. Towards an improved understanding of the origin of the low energy fine structure constant and of the mass ratios of charged fermions

As we have seen in the Lagrangian for this theory, there are two free parameters in the theory, the length scale L and the coupling constant α. These take certain values before the L-R symmetry breaking, and a different set of values after the breaking, because these values depend on whether the neutrino is Dirac (before) or Majorana (after). These values are determined by the exceptional Jordan algebra and we now make clear as to why the parameters in the Lagrangian relate to the Jordan algebra.

Before the L-R symmetry breaking the values taken by the length parameter L can be deduced from the octonionic magnitudes [70] of the fermionic states. L^2 takes the values proportional to $5/2, 3/2, 3/2, 3/2$ for the neutrino family, [anti-downquark-positron] family, [up quark - up quark] family and [positron-antidownquark] family, respectively. These families describe the four possible particle states of the unified sedenionic Lagrangian given earlier in Eqn. (90) which we reproduce here below:

$$\frac{S}{\hbar} = \int \frac{d\tau}{t_P}\, \mathcal{L} \qquad ; \qquad \mathcal{L} = \frac{1}{2} Tr \left[\frac{L_p^2}{L^2} \dot{\tilde{Q}}_1^\dagger \, \dot{\tilde{Q}}_2 \right] \tag{231}$$

The Lagrangian has an E_6 symmetry mediated by fourteen gauge bosons in sedenionic space, describing the unified force. In fact, it is more precise to not think into a split between boson and

fermion, because the variable $\tilde{Q}$ is defined as

$$\dot{\tilde{Q}}_1^\dagger = \dot{\tilde{Q}}_B^\dagger + \frac{L_p^2}{L^2}\beta_1\dot{\tilde{Q}}_F^\dagger; \qquad \dot{\tilde{Q}}_2 = \dot{\tilde{Q}}_B + \frac{L_p^2}{L^2}\beta_2\dot{\tilde{Q}}_F \tag{232}$$

Moreover the three different generations are subsumed in the definition of the sedenionic space, remembering that the latter is equivalent to three copies of the octonionic space (hence three generations). Note also the important fact that the coupling constant α does not appear explicitly, until we choose to introduce the 'would-be-gravity' aspect and the Yang-Mills aspect as a split, defined as

$$\dot{\tilde{Q}}_B = \frac{1}{L}(i\alpha q_B + L\dot{q}_B); \qquad \dot{\tilde{Q}}_F = \frac{1}{L}(i\alpha q_F + L\dot{q}_F) \tag{233}$$

By defining

$$q_1^\dagger = q_B^\dagger + \frac{L_P^2}{L^2}\beta_1 q_F^\dagger \qquad ; \qquad q_2 = q_B + \frac{L_P^2}{L^2}\beta_2 q_F \tag{234}$$

we can also express the Lagrangian as

$$\begin{aligned}\mathcal{L} &= \frac{L_P^2}{2L^2}\, Tr\left[\left(\dot{q}_1^\dagger + \frac{i\alpha}{L}q_1^\dagger\right)\times\left(\dot{q}_2 + \frac{i\alpha}{L}q_2\right)\right] \\ &= \frac{L_P^2}{2L^2}\, Tr\left[\dot{q}_1^\dagger\dot{q}_2 - \frac{\alpha^2}{L^2}q_1^\dagger q_2 + \frac{i\alpha}{L}q_1^\dagger\dot{q}_2 + \frac{i\alpha}{L}\dot{q}_1^\dagger q_2\right]\end{aligned} \tag{235}$$

The α here is actually $\alpha-$unification, and we will return to its value shortly. First though, let us see what parameter values emerge after the L-R symmetry breaking, which really should be the first time that α appears, as the coupling between the Yang-Mills aspect q_B and the would-be-gravity aspect $\dot{q}_B$ in Eqn. (233). After the L-R symmetry breaking, the L^2 parameter takes the values proportional to $3/4, 3/8, 3/8, 3/8$ for the neutrino family, antidown quark family, up quark family and the positron family, in the left-handed sector as well as in the right-handed sector.

Let us now look at Fig. (14), which describes the particle assignment after L-R symmetry breaking. On the right hand side (would-be-gravity) the RH positron (associated with $\dot{q}_F$) has a U(1) charge value $Q_{grav} = 1/3$. If the exceptional Jordan algebra for the RH positron family is diagonalised to obtain the Jordan eigenvalues, the smallest eigenvalue $(1/3 - \sqrt{3/8})$ is the projection of the RH positron on to the LH anti-down quark (associated with q_F). Keeping in view the definition of α in (233), and the earlier discussion, summarized here in Fig. (15), it is straightforward to conclude that $\ln\alpha = q = (1/3 - \sqrt{3/8}) \times 1/3$ which is the expression for α used

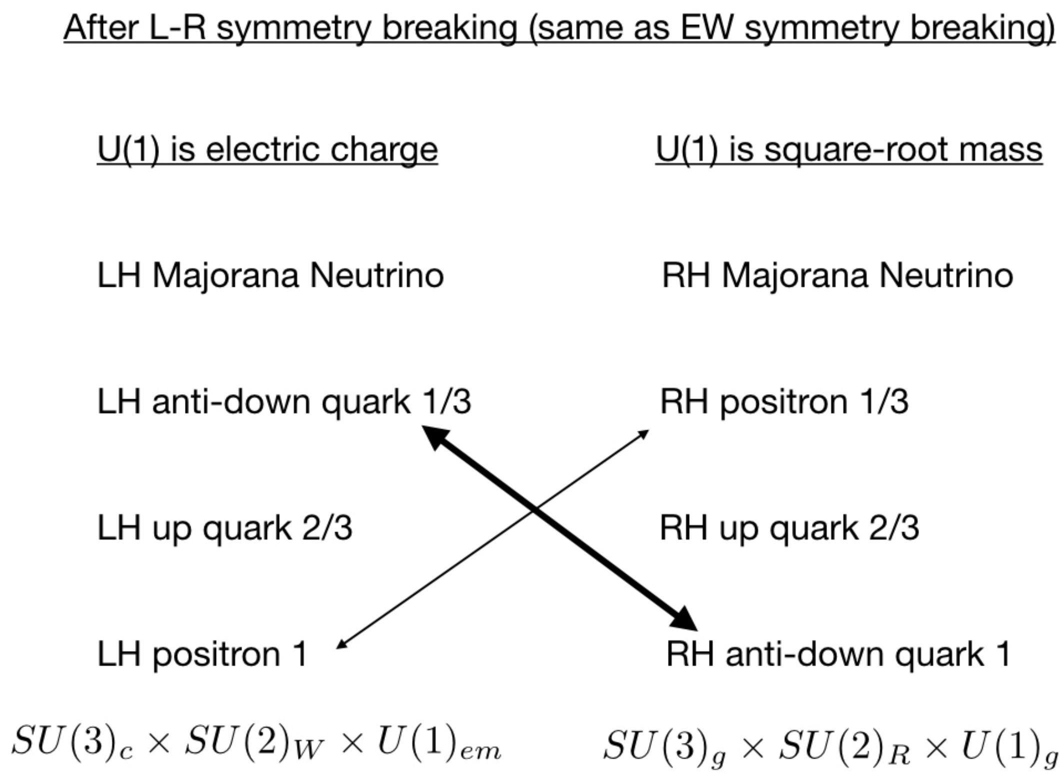


FIG. 14. Particle assignment after the L-R symmetry breaking.

while computing the low energy fine structure constant. This argument explains how and why the Lagrangian is related to the exceptional Jordan algebra. By a symmetric argument, one can infer that α_g is given by the inverse of α, i.e. $-\ln \alpha_g = (1/3 - \sqrt{3/8}) \times 1/3$, and hence that $\alpha_{unif} = 1$. This novelty results from pulling α out of the bracket in Eqn. (233) so that $1/\alpha$ appears in front of $\dot{q}_B$ and is hence identified as α_g. Gauge-gravity duality, i.e. $\alpha_g = 1/\alpha$, arises as a result of the Left-Right symmetry breaking; prior to that we do not have such a separation into gauge and gravity. That is why $\alpha_{unif} = 1$ and we only have the length parameter L. Furthermore, the gauge-gravity duality motivates the particle assignment shown in Fig. (15). This is turn helps understand why the mass ratios for the down quark family are determined using the Jordan eigenvalues for the positron family, and vice versa. Whereas the mass ratios for the up quark family come from its own set of Jordan eigenvalues. We also recall that conventionally mass is defined in quantum field theory as one of the two Casimirs associated with Poincaré symmetry, the other one being spin. This is in principle possible also in our theory with its E_6 symmetry, because as we noted earlier, the E_6 symmetry includes within itself a $SO(1,5)$ space-time symmetry. We will have a little more to say about this below, when we comment on possible connection with spinorial space-time and

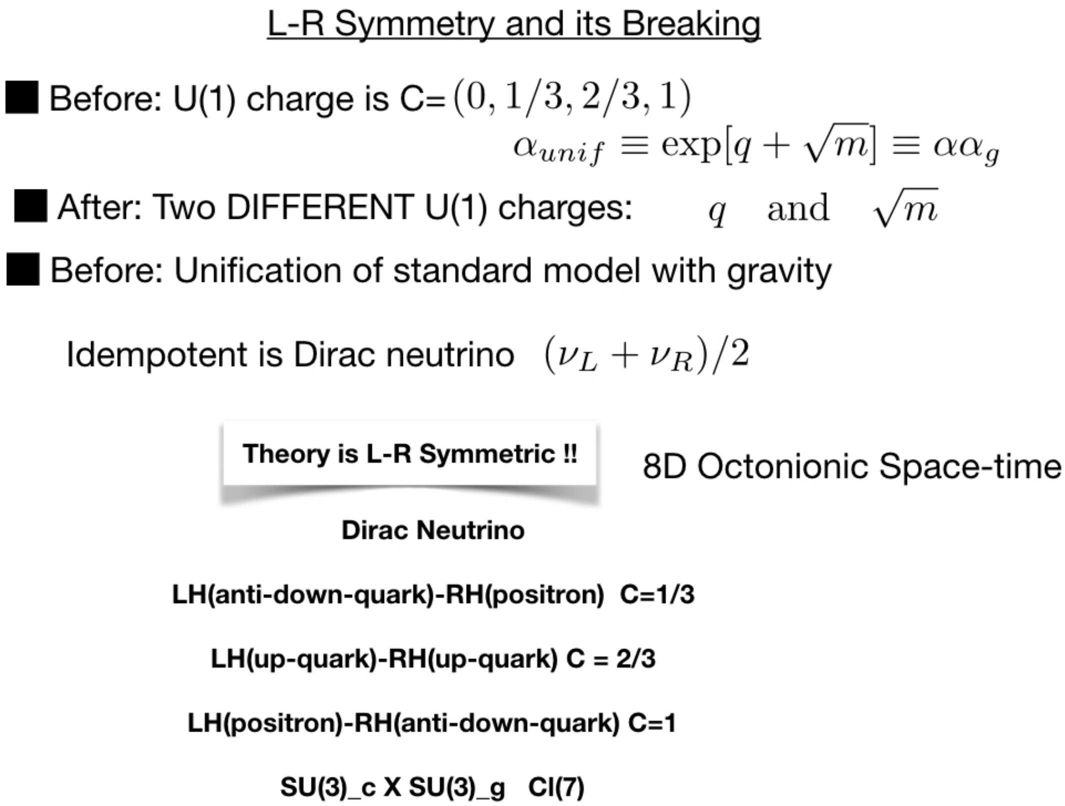


FIG. 15. Particle assignment before the L-R symmetry breaking.

twistors.

When the low energy fine-structure constant has been derived from first principles, it gives an explicit numerical relation between electric charge and the three fundamental constants of our theory, $\hbar, L_P$ and t_P. Hence the electric charge gets related to and expressed in terms of spin, via $\hbar$, and need not be considered as independent of spin. The same can be said of square-root mass, since it can also be related to the three fundamental constants of the theory. The only other free parameter is the length L which has already been related via the octonionic magnitude, to L_P. This leads us to the remarkable conclusion that there are no free quantities in this theory. The yet undetermined standard model parameters must also follow correctly from the Jordan eigenvalues; if they do not, then this cannot be the correct theory of unification.

XIII. QUANTUM THEORY AND ITS INDETERMINISM ARE EMERGENT FROM A DETERMINISTIC BUT NON-UNITARY UNDERLYING DYNAMICS; UNDERSTANDING QUANTUM NON-LOCALITY

The fundamental theory here, i.e. the generalized trace dynamics on an octonionic space, with evolution described by Connes time, is a deterministic non-unitary dynamics. It combines into one, the deterministic and the probabilistic (collapse of the wave function) aspect of quantum (field) theory. When the anti-self-adjoint part of the Hamiltonian is ignorable (i.e. when there is sub-critical entanglement amongst the degrees of freedom) the emergent dynamics is quantum field theory (without classical time). When critical entanglement takes place, non-unitary evolution becomes significant, quantum superpositions break down, and 4D classical space-time and classical material bodies emerge. Classical spacetime is 4D because the non-unitarity comes from the part of the Hamiltonian which is related to the 'internal' octonionic dimensions [the four directions e_3, e_5, e_6, e_7 along which the strong and electromagnetic force operate]. This anti-self-adjoint part of the Hamiltonian acts as a source of stochastic complex-number valued fluctuations which are then effectively equivalent to the stochastic noise in the GRW model of continuous spontaneous localisation [CSL]. Our theory provides the fundamental origin of the CSL phenomenology. This stochastic noise can be identified with the imaginary metric fluctuations proposed by Adler [65] as the source of CSL noise. Classical systems penetrate the internal directions to a depth less than Planck length. Whereas quantum systems are always in the full octonionic space. We are not predicting corrections to QFT because of these extra dimensions. Instead we are saying that 4D QFT is what it is, because spacetime is octonionic. The 'motion' identified as quantum spin in 4D is an effective description of the dynamics along the internal dimensions. We have seen earlier that in our theory spin angular momentum is defined as the canonical angular momentum corresponding to periodic motion from 4D spacetime to internal directions and back. So our assertion is that spin as defined in 4D QFT is an effective description of motion in the internal part of octonionic space, and there are no corrections coming from these higher dimensions, except the tiny Karolyhazy corrections discussed earlier in this review.

An important assumption in the CSL model, not motivated by deeper physics, is that the norm of the state vector continues to be preserved, despite the introduction of imaginary stochastic fluctuations in the quantum dynamics. This is a strong assumption, not valid in general, and in CSL it forms the basis for the emergence of the Born probability rule during wave function collapse. How then, in our theory, is norm preservation justified when non-unitary evolution

becomes significant? The answer lies in the fundamental Lagrangian given in Eqn. (231) above. This Lagrangian describes free particle motion (geodesic motion in Connes time), which during evolution preserves the length of the vector representing the particle in octonionic space. This is the origin of norm preservation and any effective emergent imaginary noise must respect this constancy of the position vector's magnitude, and hence preserve norm of the state vector.

The presence of the non-unitary sector in the Hamiltonian, in the components along the internal directions, provides an explanation for the GRW mechanism of spontaneous collapse. The deterministic unitary evolution [in the 4D spacetime part] and the reductionist non-unitary evolution [in the internal dimensions part] compete with each other, having two different characteristic time scales. Spontaneous collapse happens when the non-unitary aspect, playing the role of a stochastic noise, breaks superposition, randomly driving the system to one or the other eigenstates in the superposition, in accordance with the Born probability rule.

We can now understand how the measurement problem is resolved in our theory. Consider a detector in 'ready' state waiting to receive an incoming quantum system. In repeated trials of the same measurement, the detector, seemingly every time in the same ready state, yield different outcomes randomly, resulting in collapse of the wave function in apparent violation of the inherent linear quantum theory. From the perspective of the octonionic theory, in which rapid non-unitary variations (along the internal dimensions) are ever-present, and the successive ready states of the detector are *not* identical despite appearances. Hence different copies of the same incoming quantum system find different non-unitary configurations of the detector (along its extremely thin but non-trivial internal dimensions), resulting in collapse, with different outcomes. The octonionic theory ensures collapse in which statistically, the Born probability rule is obeyed.

A deep new feature which opens up is the role of Connes time in the emergent universe: it does not go away in a classical world, because fundamentally the theory is based in a non-commutative space, and hence Connes time as an absolute time is ever-present, as if it defines the aether. This absolute time is in addition to the 10D Minkowski space-time (which is equivalent to 8D octonionic, and contains 4D Minkowski spacetime as a subset), as if the 10D spacetime (inclusive of matter and the manifold) were in itself a mega-object evolving in Connes time, in a background of uncollapsed STM atoms. Are there multiverses (island universes, all with same physical laws) evolving in Connes time as bubbles in the background? Is Connes time to be identified with the absolute time of Newton, and this absolute time is never lost in special relativity, with the Lorentz covariant coordinate time always distinct from absolute Newton-Connes time? Thus, it is not that a particle evolves with coordinate time along a world-line. Rather, at different Connes times it is

at different points in the 4D manifold. Is the cosmic time the same as Connes-Newton time, with the expansion of the universe and the temperature of the cosmic microwave background acting as measure of cosmic time? And this aether time is over and above, and outside of coordinate time of relativity? In Newtonian mechanics we implicitly identify coordinate time with the absolute Connes time, but only Connes time flows [past, present, future, second law of thermodynamics] whereas coordinate time does not flow at all - this latter is the real aspect of quaternionic space which in itself is frozen - coordinate time is as frozen and non-flowing as three-dimensional space is. Even in general relativity, when we invoke the area theorem to conclude that the sum total of areas of the horizons increases with time, which time is it? It cannot be coordinate time or proper time, because these are themselves being dynamically defined as the black holes evolve. It has to be the Connes-Newton time, which is outside of the black hole systems.

A. Quantum non-locality

Additional internal spatial dimensions which are not compact, yet very thin, offer a promising resolution to the quantum non-locality puzzle, thereby lifting the tension with 4D special relativity. Let us consider once again Baez's cube of Fig. 3. Any of the three quaternionic spaces containing the unit element 1 can play the role of the emergent 4D classical space-time in which classical systems evolve. Let us say this classical universe is the plane $(1e_6e_1e_5)$. Now, the true universe is the full 8D octonionic universe, with the four internal dimensions being probed [only by] quantum systems. Now we must recall that these four internal dimensions are extremely thin, of the order of Fermi dimensions, and along these directions no point is too far from each other, even if their separation in the classical 4D quaternion plane is billions of light years! Consider then, that Alice at 1 and Bob at e_1 are doing space-like separated measurements on a quantum correlated pair. Whereas the event at e_1 is outside the light cone of 1, the correlated pair is always within each other's quantum wavelength along the internal directions, say the path $(1e_3e_2e_7e_1)$. The pair influences each other along this path acausally, because this route is outside the domain of 4D Lorentzian spacetime and its causal light-cone structure. The internal route is classically forbidden but allowed in quantum mechanics. This way neither special relativity nor quantum mechanics needs to be modified. It is also interesting to ask if evolution in Connes time in this 8D octonionic universe obeying generalised trace dynamics can violate the Tsirelson bound.

In this light it is worth revisiting the famous double slit interference experiment with electrons. We may conclude in our octonionic theory that after all the electron goes through only one of

the two slits, in our observed three-dimensional physical space. However, through the internal dimensions the two slits are not physically separated bur connected: in these internal dimensions the electron does go through 'both the slits' giving rise to the observed interference pattern. We will do well to note that the experimentalist needs to have the slit to be bigger in size than the incoming particle - something which would be needed only if the electron had to physically pass through one and only one of the two slits. Such a requirement on the slit size would not have arisen if the electron were to simultaneously pass through both the slits as a wave. Whereas we do know that it makes no sense to talk of a probability wave passing through both the slits, nor that the complex-valued wave function is passing through two slits in real physical space.

The following discussion explains how our theory is a natural extension of four dimensional special relativity.

B. Special relativity and the octonionic theory

Special relativity, Complex quaternions, and the algebra $\mathbb{R} \times \mathbb{C} \times \mathbb{H}$:

Consider the quaternionic four vector $\mathbf{x} = x_0e_0 + x_1e_1 + x_2e_2 + x_4e_4$ and the corresponding position four-vector for a particle in special relativity: $\mathbf{q_i} = q_0e_0 + q_1e_1 + q_2e_2 + q_4e_4$. One can define the four-metric on this Minkowski space-time whose symmetry group is the Lorentz group $SO(3,1)$ having the universal cover Spin(3,1) isomorphic to $SL(2,C)$. The complex quaternions generate the boosts and rotations of the Lorentz group SO(3,1). They can be used to obtain a faithful representation of the Clifford algebra $Cl(2)$ and fermionic ladder operators constructed from this algebra can be used to generate the Lorentz algebra $SL(2,\mathbb{C})$. Also, $Cl(2)$ can be used to construct left and right handed Weyl spinors as minimal left ideals of this Clifford algebra, and as is well known the Dirac spinor and the Majorana spinor can be defined from the Weyl spinors. Cl(2) also gives the vector and scalar representations of the Lorentz algebra. These results are lucidly described in Furey's Ph. D. thesis [25–27] as well as also in her video lecture series on standard model and division algebras https://www.youtube.com/watch?v=GJCKCss43WI&ab˙channel=CohlFureyCohlFurey

The above relation between the Clifford algebra $Cl(2)$ and the Lorentz algebra $SL(2,C)$ strongly suggests, keeping in view the earlier conclusions for $Cl(6)$ and the standard model and the octonions [25–27], that the $Cl(2)$ algebra describes the left handed neutrino and the right-handed anti-neutrino, and a pair of spin one Lorentz bosons. This is confirmed by writing the following trace dynamics Lagrangian and action on the quaternionic space-time of special relativity, thereby

generalising the relativistic particle $S = -mc\int ds$:

$$\frac{S}{C_0} = \frac{a_0}{2}\int \frac{d\tau}{\tau_{Pl}}\, Tr\left[\dot{q}_B^\dagger + i\frac{\alpha}{L}q_B^\dagger + a_0\beta_1\left(\dot{q}_F^\dagger + i\frac{\alpha}{L}q_F^\dagger\right)\right] \times \left[\dot{q}_B + i\frac{\alpha}{L}q_B + a_0\beta_2\left(\dot{q}_F + i\frac{\alpha}{L}q_F\right)\right] \quad (236)$$

where $a_0 \equiv L_P^2/L^2$. This Lagrangian is identical in form to the one studied earlier in the present paper, but with a crucial difference that it is now written on 4D quaternionic space-time, not on 8D octonionic space-time. Thus $\dot{q}_B$ and q_B have four components between them, not eight: $q_B = q_{Be2}\ e_2 + q_{Be4}\ e_4$; $\dot{q}_B = \dot{q}_{Be0}\ e_0 + \dot{q}_{Be1}\ e_1$. Similarly, the fermionic matrices have four components between them, not eight. Thus $q_F = q_{Fe2}\ e_2 + q_{Fe4}\ e_4$; $\dot{q}_F = \dot{q}_{Fe0}\ e_0 + \dot{q}_{Fe1}\ e_1$

This has far-reaching consequences. Consider first the case where we set $\alpha = 0$. The Lagrangian then is

$$\frac{S}{C_0} = \frac{a_0}{2}\int \frac{d\tau}{\tau_{Pl}}\, Tr\left[\dot{q}_B^\dagger + a_0\beta_1\dot{q}_F^\dagger\right] \times \left[\dot{q}_B + a_0\beta_2\dot{q}_F\right] \quad (237)$$

By opening up the terms into their coordinate components, the various degrees of freedom can be identified with the Higgs, the Lorentz bosons, the neutral weak isospin boson, and two neutrinos. The associated space-time symmetry is the Lorentz group $SO(3,1)$ and the associated Clifford algebra is $Cl(2)$, reminding us again of the homomorphism $SL(2,\mathbb{C}) \sim SO(3,1)$.

When α is retained, the Lagrangian describes Lorentz-weak symmetry of the leptons: electron, positron, two neutrinos of the first generation, the Higgs, two Lorentz bosons, and the three weak isospin bosons. The associated Clifford algebra is now $Cl(3)$ [related to complex biquaternions] and now all the quaternionic degrees of freedom have been used in the Lagrangian and in the construction of the particle states.. What we have here is the extension of the Lorentz algebra by an $SU(2)$, as shown in Figure (16) below, borrowed from our earlier work [50]. Now the homomorphism $SL(2,\mathbb{H}) \sim SO(5,1)$ comes into play. It would be interesting to investigate whether a quaternionic triality [107] could explain the existence of three generations of leptons. This aspect is currently under investigation.

It is now only natural that this trace dynamics be extended to the last of the division algebras, the octonions, so as to construct an octonionic special relativity. This amounts to extending the Lorentz algebra by $U(3)$, as can be inferred from Fig. 8.

Octonionic special relativity, complex octonions, and the algebra $\mathbb{R}\times\mathbb{C}\times\mathbb{H}\times\mathbb{O}$

The background space-time is now an octonionic space-time with coordinate vector $\mathbf{x} = x_0e_0 + x_1e_1 + x_2e_2 + x_4e_4 + x_3e_3 + x_5e_5 + x_6e_6 + x_7e_7$, and the corresponding eight-vector for a particle

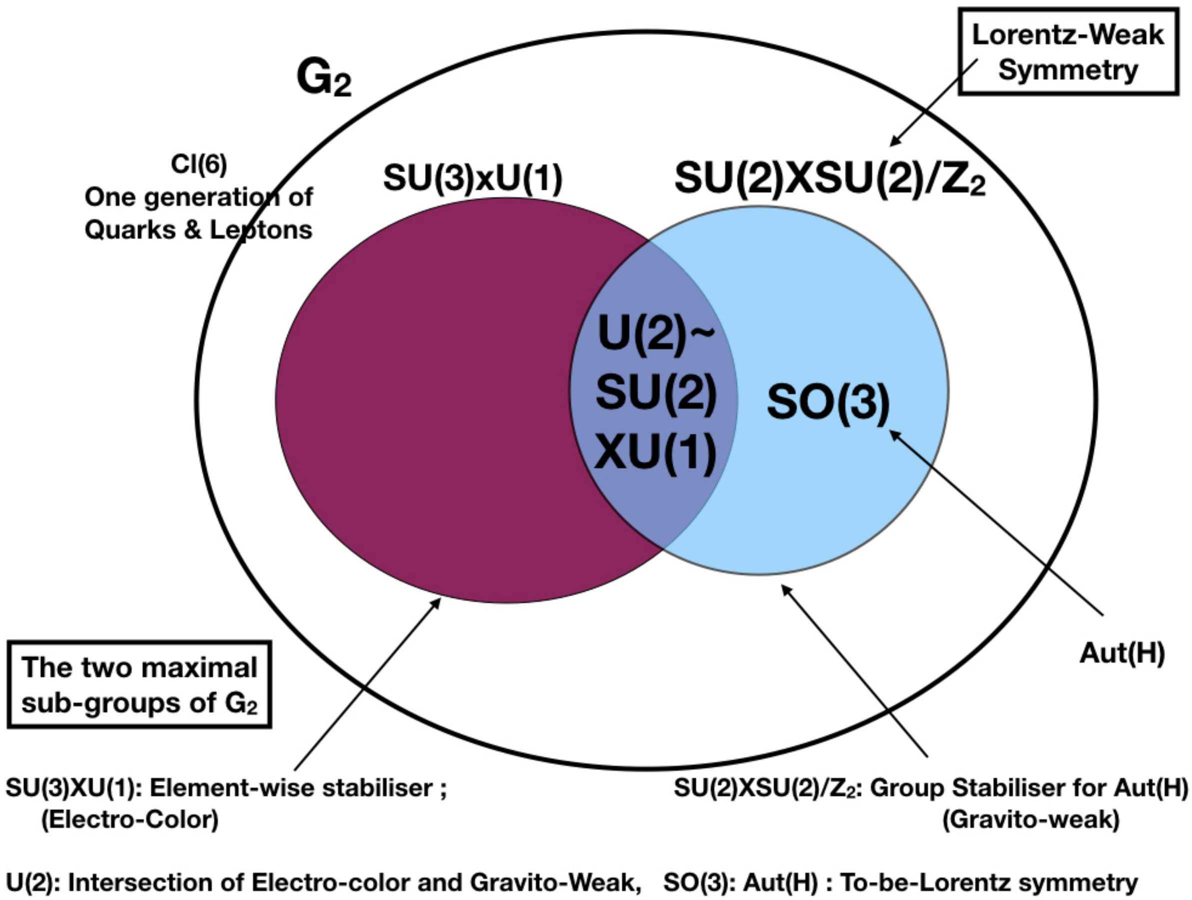


FIG. 16. The maximal sub-groups of G_2 and their intersection [From Singh [50]].

in this octonionic special relativity is $\mathbf{q_i} = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_4 e_4 + q_3 e_3 + q_5 e_5 + q_6 e_6 + q_7 e_7$. In ordinary relativity, the q_i are real numbers, but now in trace dynamics they are bosonic or fermionic matrices. The space-time symmetry group is the automorphism group G_2 of the octonions, shown in Fig. 8, along with its maximal sub-groups, which reveal the standard model along with its 4D Lorentz symmetry. The Lagrangian is the same as in (237) above, but now written on the 8D octonionic space-time. As a result, q_B and q_F have component indices (3, 5, 6, 7) whereas their time derivatives have indices (0, 1, 2, 4). This is the Lagrangian analysed in the main part of the present paper and it now includes quarks as well as leptons, along with all twelve standard model gauge bosons plus two Lorentz bosons.

We note the peculiarity that the weak part of the Lorentz-weak symmetry of the leptons, obtained by extending the Lorentz symmetry, intersects with the electr-color sector provided by $U(3) \sim SU(3) \times U(1)$. This strongly suggests that the lepton part of the weak sector can be deduced from the electro-color symmetry. This is confirmed by the earlier work of Stoica [45],

Furey [27] and our own earlier work [50].

We see that this Lagrangian is a natural generalisation of Newtonian mechanics and 4D special relativity to the last of the division algebras, the octonions, which represent a 10D Minkowski space-time because of the homomorphism $SL(2,\mathbb{O}) = SO(9,1)$.

The diagram in Fig. 17 below lists the three main steps in which the octonionic theory is developed. Current investigation is focused at the third step.

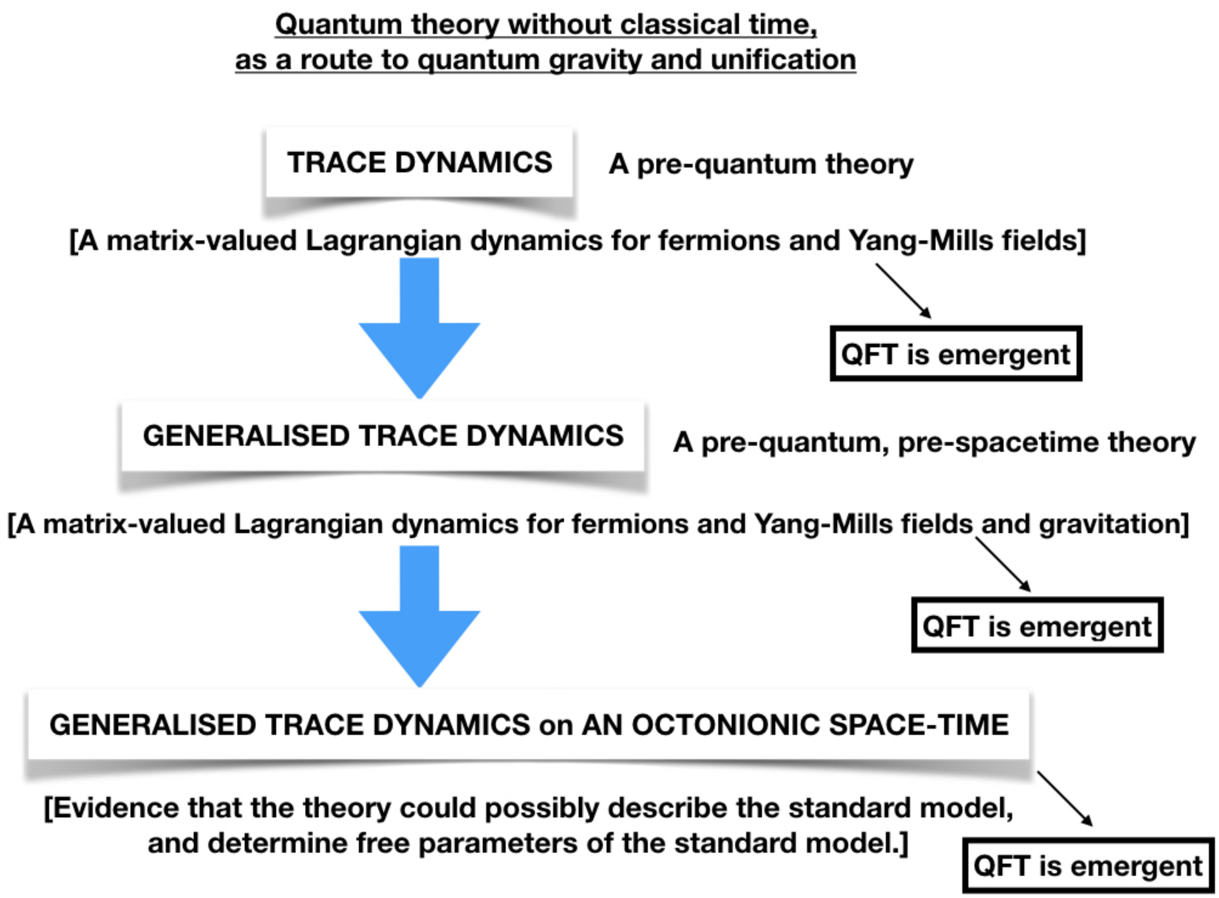


FIG. 17. The pre-space-time, pre-quantum octonionic theory in three key steps. The degrees of freedom are 'atoms of space-time-matter' [STM]. An STM atom is an elementary fermion along with all the fields that it produces. The action for an STM atom resembles a 2-brane in a 10+1 dimensional Minkowski spacetime. The fundamental universe is made of enormously many STM atoms. From here, quantum field theory is emergent upon coarse-graining the underlying fundamental theory [70].

The emergence of standard quantum field theory on a classical space-time background is a result of coarse-graining and spontaneous localisation and has been described in our earlier papers [7, 21]. Spontaneous localisation gives rise to macroscopic classical bodies and 4D classical space-time. From the vantage point of this space-time those STM atoms which have not undergone

spontaneous localisation appear, upon coarse-graining of their dynamics, as they are conventionally described by quantum field theory on a 4D classical space-time. Operationally, the transition from the action of the pre-spacetime pre-quantum theory is straightforward to describe. Suppose the relevant term in the action of the pre-theory is denoted as $\int d\tau \ \Big[Tr[T_1] + Tr[T_2] + Tr[T_3]\Big]$. Say for instance the three terms respectively describe the electromagnetic field, the action of a W boson on an electron, and the action of a gluon on an up quark. Then, the corresponding action for conventional QFT will be recovered as:

$$\int d\tau \ \Big[Tr[T_1] + Tr[T_2] + Tr[T_3]\Big] \quad \longrightarrow \quad \int d\tau \ \int d^4x \Big[[T_{1QFT}] + [T_{2QFT}] + [T_{3QFT}]\Big] \tag{238}$$

The trace has been replaced by the space-time volume integral, and each of the three terms have correspondingly been replaced by the conventional field theory actions for the three cases: conventional action for the electromagnetic field, for the W boson acting on the electron, and for the gluon acting on the up quark. In this way, QFT is recovered from the pre-theory.

However, by starting from the pre-theory, we can answer questions which the standard model cannot answer. We know now why the standard model has the symmetries it does, and why the dimensionless free parameters of the standard model take the values they do. These are fixed by the algebra of the octonions which defines the 8D octonionic space-time in the pre-theory. While this is work in progress, it provides a promising avenue for understanding the origin of the standard model and its unification with gravitation.

XIV. THE UNIVERSE BEFORE, AND AFTER, ELECTRO-WEAK SYMMETRY BREAKING: COMPACTIFICATION WITHOUT COMPACTIFICATION; ONLY CLASSICAL SYSTEMS LIVE IN 4D, QUANTUM SYSTEMS ARE ALWAYS IN 10D

In the present theory, the electroweak symmetry breaking results from the separation of the electro-color sector from the Lorentz-weak sector. This symmetry breaking is also the Left-Right symmetry breaking, and can be represented as follows, keeping in mind the symmetry group E_6:

$$\begin{aligned}&\Big[SU(3)_c \times U(1)_{em} \oplus SU(3)_{grav} \times U(1)_{grav}\Big] \otimes \Big[SU(2)_L \oplus SU(2)_R\Big] \longrightarrow \\ &\Big[SU(3)_c \times U(1)_{em} \otimes SU(2)_L\Big] \oplus \Big[SU(3)_{grav} \times U(1)_{grav} \otimes SU(2)_R\Big]\end{aligned} \tag{239}$$

The first bracket is the standard model, while the second bracket is would-be-gravity. The Higgs mechanism and 4D classical general relativity are expected to emerge from the second bracket. The emergent universe after symmetry breaking is a four-dimensional classical space-time geometry, whereas the quantum systems obeying standard model internal symmetries continue to remain in 8D octonionic space.

This compactification without compactification is a very major advantage over string theory, because an effective compactification is achieved dynamically [via spontaneous localisation] without artificially curling the extra dimensions. The particle content of the theory has been constructed from an algebraic vacuum (this being the neutrino) using Clifford algebra / geometric algebra. The particle content is not constructed by making a Fock space around the Minkowski vacuum. These differences from string theory explain why string theory fails as a theory of unification, whereas the octonionic theory succeeds. And yet there are similarities between the two theories: both describe the dynamics of an extended object in 10D Minkowski spacetime. The following are the key differences from string theory: (i) In the octonionic theory, evolution is in Connes time, (ii) the Hamiltonian in general has an anti-self-adjoint part which facilitates compactification without compactification, (iii) the vacuum is an algebraic vacuum in a Clifford algebra, and spinorial particle states are obtained as excitations of the neutrino, (iv) dynamical laws are those of trace dynamics, not of quantum field theory; QFT is emergent.

In this, given the modified quantum mechanics that we have here (generalised trace dynamics with a non-self-adjoint Hamiltonian), the O-theory has a distinct advantage over all the interpretations of quantum mechanics which do not modify quantum theory. The various interpretations offer an explanation of the quantum-to-classical transition while always staying in four-dimensional spacetime. If standard quantum theory is implemented in ten dimensions, it can descend to four dimensions only by artificially curling the extra dimensions and doing so destroys predictability. When there is compactification without compactification, predictability is not lost. This is one instance where modified quantum mechanics has a distinct advantage our quantum interpretations such as many worlds.‘

The universe prior to the L-R (also electroweak) symmetry breaking is extremely different from what we are accustomed to thinking of it as - a four dimensional universe obeying FRW cosmology. In the O-theory, we instead have a higher dimensional universe (8D octonionic = 10D Minkowski) evolving in Connes time. All interactions, including gravity, are unified before this symmetry breaking, and the neutrino is a Dirac neutrino. It is not as if the particles were massless before the symmetry breaking and acquired a non-zero mass after symmetry breaking. Rather,

after the symmetry breaking, the right-hand-sector is coupled to the left-hand-sector via the Higgs mechanism, effectively creating a theory which is equivalent to 'acquiring mass'. Non-zero mass was always there to begin with [and this does not violate gauge invariance in the octonionic theory] being given by the square-root mass numbers which express mass as a fraction of Planck mass. The expansion of the octonionic universe, starting from the big bang event, lowers the masses as a function of Connes time, until the symmetry breaking happens, when the masses are frozen to the values we observe in the laboratory today. This is part of the solution to the hierarchy problem - the puzzle of the much lower EW symmetry breaking energy scale (as compared to Planck scale).

But what exactly causes the end of the 8D cosmic expansion and the breaking of the L-R symmetry at this particular energy scale? The answer lies in spontaneous localisation, and emergence of classical density perturbations. The phase prior to the L-R symmetry breaking is the equivalent of de Sitter cosmic inflation; the cooling which accompanies the expansion permits increasingly enhanced entanglement until critical entanglement is reached and a quantum-to-classical transition takes place, leading to the L-R symmetry breaking. The universe comes to be dominated by classical objects, these possibly being Planck mass primordial black holes, possessing a scale-invariant spectrum of density perturbations. The scale of the symmetry breaking is set by a competition between the following two processes: (i) a 'repulsive gravity' like cosmic acceleration akin to the presently observed accelerating universe (more on this in the section on MOND below), and (ii) an attractive gravity like local effect in the vicinity of the critically entangled degrees of freedom which undergo spontaneous localisation and form classical objects. When the latter begin to dominate over the former, the symmetry breaking happens as a result of a phase transition, and this is what sets the electroweak scale. In the symmetry broken phase, classical objects dominate over quantum systems, and the universe becomes 4D classical.: gravitation as we know it emerges from would-be-gravity, and the expansion of the universe transforms from de Sitter like to a power law expansion in FRW cosmology. In effect, we have cosmic inflation ending at the EW scale, and particle masses are frozen to the values they have at the epoch of this phase transition.

We end this section with two further remarks. Firstly, we do not expect any corrections to the quadratic Lagrangian proposed here, at higher energies / curvatures. The reason is that this Lagrangian very closely mirrors properties of the octonionic space in which the aikyons live. In particular, there is a close connection with the octonionic projective plane OP^2. Unlike for the other three division algebras, which can have a projective space of any number of dimensions, the octonions can only have a projective line or a projective plane. Octonionic projective space of dimension $n = 3$ of higher is not allowed because that would make the algebra associative. The

quadratic Lagrangian here is related to OP^2, and since $n > 2$ is not permitted, a higher order correction to the Lagrangian is not possible.

Our second remark is a word of caution. Although we talk of emergent classical spacetime which is 4D, we recall that with E_6 symmetry, the emergent spacetime after symmetry breaking is actually 6D Minkowski (not 4D) and has an $SO(1,5)$ symmetry. This is related to the complex biquaternions and the Clifford algebra $Cl(3)$ and is the ideal place for describing gravi-weak unification. Thus it could well be that our 4D classical space-time is the large part of 6D Minkowski spacetime, with the extra two spatial dimensions being very thin, and describing the weak interaction as the (quantum mechanical) internal counterpart of 4D gravitation.

XV. PREDICTIONS FOR EXPERIMENTS, AND THE PHENOMENOLOGY OF E_6

Our theory has no free parameters and is falsifiable. If the predicted Lorentz boson is ruled out by experiments, our theory will be ruled out. Also, our theory predicts that only other new particles to be discovered, apart from the Lorentz boson, are sterile neutrinos and a $U(1)$ gravitational boson. Thus the discovery of additional other new particles, say a fourth generation of fermions, will also rule out our theory. Elsewhere, we have predicted the experimentally testable Karolyhazy length uncertainty relation as a consequence of our theory. This says that if a device is used to measure a length L, there will be a minimum uncertainty ΔL, given by the Karolyhazy relation [74, 108–113]

$$(\Delta L)^3 \sim L_P^2 \, L \tag{240}$$

A dedicated experiment is planned to test this relation [114]. We also note that this relation implies holography: quantum information in a region of size L grows as area of the region's boundary, not as the region's volume. This is because if ΔL is the smallest possible linear extent of a cell with one unit of information, then it follows from this relation that $L^3/(\Delta L)^3 \sim L^2/L_P^2$ which of course is holography. Note that the minimum of length in our theory is not L_P, but the much larger ΔL given by the above relation. A holographic theory of quantum gravity must necessarily predict and satisfy this holographic relation.

We have also predicted the unification of gravity and the weak interaction. This implies a variation in the value of G_N which must be looked for at small scales [55, 56]. Our theory also predicts the phenomenon of spontaneous localisation, currently being tested in the laboratory [115, 116]. We predict a specific model of Continuous Spontaneous Localisation [CSL] in the class

of objective collapse models. The collapse rate λ is given by $(L_P/L)^3\ \tau_P^{-1}$ where L is the Compton wavelength of the proton. Numerically, this is of the order of 10^{-17} s^{-1} which agrees with the value proposed by the Ghirardi-Rimini-Weber-Pearle theory. The collapse rate in our theory grows as cube of the mass in the system; hence this is not the mass-proportional CSL model. We are currently investigating the CSL noise spectrum predicted by the underlying aikyon theory - this will aid comparison of the aikyon theory versus experiment.

In addition, we have predicted the testable novel phenomenon of spontaneous localisation in time [19]. Thus, our theory makes several predictions, testable with current technology, which can be used to confirm or rule out the theory. We have also explained the remarkable fact the the Kerr-Newman black hole has the same gyromagnetic ratio as the electron [21].

It would also be interesting to explore if there is a left-over Lorentz radiation background [made of Lorentz bosons] from the beginnings of the universe, analogous to the cosmic microwave background. And whether this radiation could have some role to play as dark energy? We have in fact recently suggested, based on our theory, that dark energy is a large scale quantum gravitational phenomenon [117].

The left-right symmetric model explains the strong CP problem, origin of matter-antimatter asymmetry, the vanishing of Higgs coupling prior to symmetry breaking, the mass of the neutrino through see-saw mechanism, and predicts a right-handed sterile neutrino which is a dark matter candidate [83, 85, 86].

We also predict the neutrino to be a Majorana particle, and hence that neutrinoless double beta decays occurs in nature.

XVI. A POSSIBLE CONNECTION WITH TWISTOR SPACES AND SPINORIAL SPACE-TIME

We would also like to draw a correspondence between the spinor spacetime as described by Penrose and the quaternionic spacetime we are discussing about. In quantum field theory a spinor field undergoes active transformation as:

$$\psi^{'}(x) = \lambda_s \psi(\lambda^{-1} x) \tag{241}$$

Here λ_s is the Lorentz transformation in irreducible representation whereas λ^{-1} is the Lorentz transformation in vector representation. The two different representations of Lorentz algebra are

required because all fields are defined at a spacetime point in the manifold. Why should we use spinor representation of Lorentz algebra for the field while using vector representation for the spacetime point? It seems that quantum field theory should be done on a spinor spacetime from which our classical spacetime emerges. Penrose and Rindler [118] talk about a spinor spacetime from which the classical spacetime emerges. This spinor spacetime can be obtained by representing null vectors on a Riemann sphere and then taking a stereographic projection from the Riemann sphere to a complex Argand plane. We can span our Minkowski spacetime using points on the complex plane instead of using 4 null-like vectors. We can talk of spacetime spinors using quaternions as well, therefore the quaternionic spacetime that we are discussing from $Cl(2)$ might be Penrose's spinor spacetime. Similarly the 8 dimensional twistor spacetime which is a subset of the projective space $\mathbf{CP^3}$ (space of lines passing through the origin in $\mathbf{C^4}$) might be similar to the octonionic spacetime mentioned in this paper. The octonionic spacetime will have four more internal dimensions for explaining quarks. Further investigation is under progress.

One should explore also the possible existence of the 'square-root' of the Dirac equation, by which we mean writing the Dirac equation, not on the vectorial Minkowski spacetime, but on a spinorial spacetime, where the source term is not the mass of the fermion, but the square root of mass. This will help understand why square-root mass ratios are what emerge naturally from the exceptional Jordan algebra, not the mass ratios themselves. Squaring this spinorial Dirac equation should give the usual Dirac equation, with mass as its source. The spinorial spacetime would then provide a 'square-root' representation of Minkowski spacetime, and we will be able to present square-root mass as the Casimir of such a spinor version of Poincaré symmetry.

XVII. IS THERE A RELATION WITH MODIFIED NEWTONIAN DYNAMICS (MOND)?

MOND is an alternative to dark matter. Instead of proposing that the dynamical anomaly of galactic rotation curves is due to the presence of additional invisible matter, it is proposed that on sufficiently large distance scales, where gravitational acceleration falls below a critical value a_0, the law of gravitation departs from Newton's. Roughly put, in MOND, the circular orbital acceleration v^2/R outside a mass M is given as usual by Newton's GM/R^2, so long as the acceleration exceeds a_0. This gives the well-known Keplerian fall-off $V^2 \sim 1/R$, which is contradicted by the flat galaxy rotation curves. As is known from observations, whenever the observed orbital acceleration falls below a_0, the velocity curve stays flat thereafter. In 1983, Milgrom proposed that in the deep MOND regime, where the acceleration is much below a_0, the law of gravitation changes, so that

the orbital acceleration is now given by $v^2/R \sim (GMa_0)^1/2/R$ which explains the flatness of the rotation curve. This phenomenological modification of Newtonian gravitation at large distances is quite successful in explaining observations. Clearly a modification to general relativity is implied at large distances, including at the cosmological Hubble scale. Curiously, a_0 is numerically of the order of the cosmic acceleration c$/H_0$. A mysterious coincidence or a pointer to a relativistic theory underlying MOND? Until recently, the main and justified criticism of MOND was that there is no relativistic extension of the theory which can account for structure formation and in particular the CMB anisotropies. Something at which cold dark matter is highly successful. This may have changed last year, when two physicists constructed RMOND [119], an action principle based phenomenological relativistic extension of MOND, which explains CMB data.

What could be the fundamental origin of MOND? This is where the octonionic theory comes in. What caught the present author's attention is the acceleration being proportional to square-root of M, instead of M, in the MOND regime. In the O-theory as well, the would-be-gravity theory $SU(3)_g \times SU(2)_R \times U(1)_g$ has as its charge square-root of m, rather than m, where m is the mass of the elementary particle. In the unbroken L-R symmetric regime, the interaction strength goes as $sqrtm$. When L-R symmetry is broken, squaring of would-be-gravity is enabled, GR and Newtonian gravitation emerge, and the interaction strength goes as m. Where and how does a_0 enter the picture? We will identify a_0 with cosmic acceleration at the corresponding cosmic epoch (making it epoch dependent!). The L-R symmetry breaking [same as electro-weak symmetry breaking] is caused by spontaneous localisation of classical matter perturbations (primordial black holes?) as a result of which the emergent gravitational acceleration in the vicinity of compact objects exceeds the (pre symmetry breaking) cosmic acceleration a_0. This would be the origin of MOND. In the vicinity of compact objects, where acceleration exceeds a_0, the square of would-be-gravity, i.e. GR and Newtonian gravity hold. As for instance in the solar system and near stars and black holes. However, in low density regions, where acceleration is below the cosmic acceleration a_0, the unbroken would-be-gravity law holds, where acceleration is proportional to square root mass, not mass. If this line of thinking were to be correct, the octonionic theory could explain the fundamental origin of MOND. The critical acceleration a_0 then serves as the order parameter for a phase transition: the L-R symmetry breaking. Could it be that the $U(1)_g$ of would-be-gravity is the sought after dark energy. i.e. dark photons? In a universe made only of matter, all particles have like charge $\sqrt{m}$, and the U(1) vector interaction is repulsive. MOND would then be more fundamental than Newtonian gravitation, with the latter becoming square of MOND! MOND is then non-relativistic limit of would-be-gravity.

In what way might we think of Newtonian gravitation as the square of MOND? To get some insight, let us write the acceleration a in a circular orbit as $a = [GM/R^2](1 + \beta(R)]$. The function $\beta(R)$ depends on a and goes to zero for $a]gga_0$, thus recovering Newtonian gravitation. For a in the vicinity of a_0 we approximate the last bracket to a_0/a, thus yielding MOND. At large cosmic distances, relativistic effects become important. GM/R^2 is replaced by its GR counterpart, and $(1 + \beta)$ becomes the MOND induced modification of GR. It is important to ask if dark energy is a manifestation of this MOND induced modification. Were this to be so, we will have a common origin cause for flat galaxy rotation curves and for cosmic acceleration, without dark matter or dark energy, and because GR and Newtonian gravity are limiting cases of a more general law of gravity. On the clusters scale MOND will need warm dark matter such as sterile neutrinos, or dark baryons. What then is this more general law of gravity? Which we demand must come from first principles. The Left-Right symmetric octonionic theory proposes $SU(3)_g \times SU(2)_R \times U(1)_g$ as would-be-gravity, or square-root-gravity, this is the right-hand counterpart of the broken L-R symmetric theory, whose left-handed counterpart is the standard model. The gauge theory of would-be-gravity on an 8D octonionic space-time is proposed as the more general law of gravity, which explains the origin of the critical acceleration a_0 and the emergence of GR, MOND, and Newtonian gravity as special cases.

Cosmology and the scale a_0: In the L-R symmetric theory, the very early universe undergoes an inflationary expansion having a time-dependent cosmic acceleration $a_0(t)$. This inflationary expansion is halted (and converted to a power law expansion) when significant seeding of density perturbations causes a quantum-to-classical transition, L-R symmetry breaking, and emergence of 4D classical spacetime. The gravitational acceleration in the vicinity of the seeded relativistic density perturbations exceeds the then a_0, and GR emerges from would-be-gravity as its square. MOND is the transition zone between would-be-gravity and Newton/GR. Thus would-be-gravity can seed the scale-invariant matter perturbations whose effect is seen in the CMB (hence relativistic MOND). In today's universe, away from compact objects, would-be-gravity (relativistic MOND) dominates because accelerations are smaller than a_0 and tend to the current cosmic a_0. In this deep MOND regime there is space-time scale invariance, and the universe tends to de Sitter. Would-be-gravity when squared yields GR at high accelerations and the condensation of $SU(2)_L$ and $SU(2)_R$ into 4D/6D spacetime geometry is indicated. $SU(2)_L$ mediated on small scales by heavy weak bosons is the weak interaction. The electro-weak symmetry breaking is in reality an L-R symmetry breaking same as: QCD Color + $U(1)_{em}$ breaks from Weak $SU(2)_L$. Grav Color + $U(1)_g$ breaks from $SU(2)_R$. It appears that if we do cosmology with the L-R symmetric octonionic

theory and its emergent approximations, all could be well without cold dark matter and without cosmological constant as dark energy. In this theory the cosmological constant (zero point energy of vacuum) is strictly zero. Cosmic acceleration is caused by dark photons obeying $U(1)_g$ symmetry.

While discussing the Chamseddine-Connes spectral action principle for the Dirac operator including the Yang-Mills fields, we had noted that squaring the Dirac operator yields not only the Einstein-Hilbert action but also an action term for conformal gravity. This is a hint that this term maybe related to relativistic MOND, and might be competing with the action term of the Einstein-Hilbert action, and the latter (including its Newtonian limit) could be dominant near compact objects where the acceleration exceeds the (critical) cosmic acceleration. In low acceleration regions conformal gravity could dominate, providing the relativistic version of MOND. This is under investigation.

XVIII. CONCLUSIONS, AND OUTLOOK FOR FURTHER WORK

We have the complete scaffolding for a promising theory of unification; now details need to be built in to see if the theory fulfils its initial promise. We have a matrix-valued Lagrangian dynamics which is a pre-spacetime, pre-quantum theory. The motivation for the theory comes from quantum foundations; to construct a reformulation of quantum field theory which does not depend on classical time. We successfully arrive at such a theory, and remarkably it has strong overlap with string theory in ten dimensions. However there are crucial differences from string theory, as enunciated above, which help us understand very clearly as to why string theory fails as a theory of unification, whereas our octonionic theory succeeds. We successfully predict the mass ratios of charged fermions, and the low energy fine structure constant, to an accuracy good enough for us to investigate this theory further.

The following are some of the important questions which still need to be answered in this theory, listed in no particular order:

- Predict the masses, both for active neutrinos and sterile neutrinos.
- Predict the QCD coupling constant and the weak coupling constant.
- Predict the CKM quark mixing matrix.
- Understand the origin of matter-antimatter asymmetry.

- Understand the origin of general relativity and relativistic MOND, from the underlying would-be-gravity.

- Construct the 'square-root' Dirac equation, i.e. the Dirac equation written on a spinorial spacetime, with square-root mass as source.

- Understand the connection of the octonionic theory with the twistor theory.

- What is the possible role of torsion in this theory?

- Understand and develop the gravi-weak theory on an SO(1,5) six dimensional spacetime.

- Predict the CSL noise spectrum which experimentalists can then test for.

- Develop the physics of the phase transition leading to electro-weak and Left-Right symmetry breaking.

- Derive the Higgs mass and electro-weak symmetry breaking scale from first principles.

- Derive the Higgs mechanism as a consequence of the coupling of would-be-gravity with the standard model

- Understand the significance and implications of Connes time.

- Understand quark confinement and asymptotic freedom as properties of the octonionic space and the accompanying Lagrangian

- Understand C, P and T symmetries, and their violation in certain situations, as a consequence of the octonionic space and the accompanying Lagrangian

- Understand interactions in the octonionic theory in terms of matrix-valued collisions, and entanglement

  The octonionic theory has no free parameters. It will hence be falsified if it cannot give a satisfactory answer to all these questions listed above.

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