The characteristic equation of the exceptional Jordan algebra: its eigenvalues, and their possible connection with mass ratios of quarks and leptons

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Abstract

The exceptional Jordan algebra [also known as the Albert algebra] is the finite dimensional algebra of 3x3 Hermitean matrices with octonionic entries. Its automorphism group is the exceptional Lie group $F_4$. These matrices admit a cubic characteristic equation whose eigenvalues are real and depend on the invariant trace, determinant, and an inner product made from the Jordan matrix. Also, there is some evidence in the literature that the group $F_4$ could play a role in the unification of the standard model symmetries, including the Lorentz symmetry. The octonion algebra is known to correctly yield the electric charge values (0, 1/3, 2/3, 1) for standard model fermions, via the eigenvalues of a $U(1)$ number operator, identified with $U(1)_{em}$. In the present article, we use the same octonionic representation of the fermions to compute the eigenvalues of the characteristic equation of the Albert algebra, and compare the resulting eigenvalues with the known mass ratios for quarks and leptons. We find that the ratios of the eigenvalues correctly reproduce the [square root of the] known mass ratios for up, charm and top quark. We also propose a diagrammatic representation of the standard model bosons, Higgs and three fermion generations, in terms of the octonions, exhibiting an $F_4$ symmetry. We motivate from our Lagrangian as to why the eigenvalues computed in this work could bear a relation with mass ratios of quarks and leptons.

I. INTRODUCTION

The possible connection between division algebras, exceptional Lie groups, and the standard model has been a subject of interest for many researchers in the last few decades [1–23]. Our own interest in this connection stems from the following observation [24]. In the pre-geometric, pre-quantum theory of generalised trace dynamics, the definition of spin...
requires 4D space-time to be generalised to an 8D non-commutative space. In this case, an octonionic space is a possible, natural, choice for further investigation. We found that the additional four directions can serve as ‘internal’ directions and open a path towards a possible unification of the Lorentz symmetry with the standard model, with gravitation arising only as an emergent phenomenon. Instead of the Lorentz transformations and internal gauge transformations, the symmetries of the octonionic space are now described by the automorphisms of the octonion algebra. Remarkably enough, the symmetry groups of this algebra, namely the exceptional Lie groups, naturally have in them the desired symmetries [and only those symmetries, or higher ones built from them] of the standard model, including Lorentz symmetry, without the need for any fine tuning or adjustments. Thus the group of automorphisms of the octonions is $G_2$, the smallest of the five exceptional Lie groups $G_2, F_4, E_6, E_7, E_8$. The group $G_2$ has two intersecting maximal sub-groups, $SU(3)\times U(1)$ and $SU(2)\times SU(2)$, which between them account for the fourteen generators of $G_2$, and can possibly serve as the symmetry group for one generation of standard model fermions. The complexified Clifford algebra $Cl(6,C)$ plays a very important role in establishing this connection. In particular, motivated by a map between the complexified octonion algebra and $Cl(6,C)$, electric charge is defined as one-third the eigenvalue of a $U(1)$ number operator, which is identified with $U(1)_{em}$ [3, 5].

Describing the symmetries $SU(3)\times U(1)$ and $SU(2)\times SU(2)$ of the standard model [with Lorentz symmetry now included] requires two copies of the Clifford algebra $Cl(6,C)$ whereas the octonion algebra yields only one such independent copy. It turns out that if boundary terms are not dropped from the Lagrangian of our theory, the Lagrangian describes three fermion generations, with the symmetry group now raised to $F_4$. This admits three intersecting copies of $G_2$, with the $SU(2)\times SU(2)$ in the intersection, and a Clifford algebra construction based on the three copies of the octonion algebra is now possible [25]. Attention thus shifts to investigating the connection between $F_4$ and the three generations of the standard model.

$F_4$ is also the group of automorphisms of the exceptional Jordan algebra [11, 26, 27]. The elements of the algebra are 3x3 Hermitean matrices with octonionic entries. This algebra admits an important cubic characteristic equation with real eigenvalues. Now we know that the three fermion generations differ from each other only in the mass of the corresponding fermion, whereas the electric charge remains unchanged across the generations. This
motivates us to ask: if the eigenvalues of the $U(1)$ number operator constructed from the octonion algebra represent electric charge, what is represented by the eigenvalues of the exceptional Jordan algebra? Could these eigenvalues bear a connection with mass ratios of quarks and leptons? This is the question investigated in the present paper. Using the very same octonion algebra which was used to construct a state basis for standard model fermions, we calculate these eigenvalues. Remarkably, the eigenvalues are very simple to express, and bear a simple relation with electric charge. We comment on how they could relate to mass ratios. In particular we find that the ratios of the eigenvalues match with the square root of the mass ratios of up quark, charm, and the top. [These eigenvalues are invariant under algebra automorphisms.]

Subsequently in the paper we propose a diagrammatic representation, based on octonions and $F_4$, of the fourteen gauge bosons, and the $(8x2)x3 = 48$ fermions of three generations of standard model, along with the four Higgs. We attempt to explain why there are not three generations of bosons, and re-express our Lagrangian in a form which explicitly reflects this fact. We hint at how this Lagrangian might directly lead to the characteristic equation of the exceptional Jordan algebra, and reveal why the eigenvalues might be related to mass.

II. EIGENVALUES FROM THE CHARACTERISTIC EQUATION OF THE EXCEPTIONAL JORDAN ALGEBRA

The exceptional Jordan algebra [EJA] $J_3(O)$ is the algebra of 3x3 Hermitean matrices with octonionic entries [12, 21, 22, 26]

$$X(\xi, x) = \begin{bmatrix} \xi_1 & x_3 & x_2 \\ x_3^* & \xi_2 & x_1 \\ x_2 & x_1^* & \xi_3 \end{bmatrix}$$

(1)

It satisfies the characteristic equation [12, 21, 22]

$$X^3 - Tr(X)X^2 + S(X)X - Det(X) = 0; \quad Tr(X) = \xi_1 + \xi_2 + \xi_3$$

(2)
which is also satisfied by the eigenvalues $\lambda$ of this matrix

$$\lambda^3 - Tr(X)\lambda^2 + S(X)\lambda - Det(X) = 0$$

(3)

Here the determinant is

$$Det(X) = \xi_1\xi_2\xi_3 + 2Re(x_1x_2x_3) - \sum_{i=1}^{3} \xi_ix_i^*$$

(4)

and $S(X)$ is given by

$$S(X) = \xi_1\xi_2 - x_3x_3^* + \xi_2\xi_3 - x_1x_1^* + \xi_1\xi_3 - x_2x_2$$

(5)

The diagonal entries are real numbers and the off-diagonal entries are (real-valued) octonions. A star denotes an octonionic conjugate. The automorphism group of this algebra is the exceptional Lie group $F_4$. Because the Jordan matrix is Hermitean, it has real eigenvalues which can be obtained by solving the above-given eigenvalue equation.

In the present Letter we suggest that these eigenvalues carry information about mass ratios of quarks and leptons of the standard model, provided we suitably employ the octonionic entries and the diagonal real elements to describe quarks and leptons of the standard model. Building on earlier work [3, 4, 19] we recently showed that the complexified Clifford algebra $Cl(6, C)$ made from the octonions acting on themselves can be used to obtain an explicit octonionic representation for a single generation of eight quarks and leptons, and their anti-particles. In a specific basis, using the neutrino as the idempotent $V$, this representation is as follows [3, 24]. The $\alpha$ are fermionic ladder operators of $Cl(6, C)$ (please see
Eqn. (34) of [24]).

\[ V = \frac{i}{2} e_7 \quad [V_\nu \text{ Neutrino}] \]

\[
\alpha_1^\dagger V = \frac{1}{2} (e_5 + ie_4) \times V = \frac{1}{4} (e_5 + ie_4) \quad [V_{\text{ad1}} \text{ Anti - down quark}] \\
\alpha_2^\dagger V = \frac{1}{2} (e_3 + ie_1) \times V = \frac{1}{4} (e_3 + ie_1) \quad [V_{\text{ad2}} \text{ Anti - down quark}] \\
\alpha_3^\dagger V = \frac{1}{2} (e_6 + ie_2) \times V = \frac{1}{4} (e_6 + ie_2) \quad [V_{\text{ad3}} \text{ Anti - down quark}] \\
\]

\[
\alpha_3^\dagger \alpha_2^\dagger V = \frac{1}{4} (e_4 + ie_5) \quad [V_{u1} \text{ Up quark}] \\
\alpha_1^\dagger \alpha_3^\dagger V = \frac{1}{4} (e_1 + ie_3) \quad [V_{u2} \text{ Up quark}] \\
\alpha_2^\dagger \alpha_1^\dagger V = \frac{1}{4} (e_2 + ie_6) \quad [V_{u3} \text{ Up quark}] \\
\alpha_3^\dagger \alpha_2^\dagger \alpha_1^\dagger V = -\frac{1}{8} (i + e_7) \quad [V_{e+} \text{ Positron}] \\
\]

The anti-particles are obtained from the above representation by complex conjugation [3].

In the context of the projective geometry of the octonionic projective plane \( \mathbb{O}P^2 \) it has been shown by Baez [14] that upto automorphisms, projections in EJA take one of the following four forms, having the respective invariant trace 0, 1, 2, 3.

\[
p_0 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \quad (7) \\
p_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \quad (8) \\
p_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \quad (9) \\
p_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \quad (10)
\]
Since it has earlier been shown by Furey [3] that electric charge is defined in the division algebra framework as one-third of the eigenvalue of a $U(1)$ number operator made from the generators of the $SU(3)$ in $G_2$, we propose to identify the trace of the Jordan matrix with the sum of the charges of the three identically charged fermions across the three generations. Thus the trace zero Jordan matrix will have diagonal entries zero, and will represent the (neutrino, muon neutrino, tau-neutrino). The trace one Jordan matrix will have diagonal entries $(1/3, 1/3, 1/3)$ and will represent the (anti-down quark, anti-strange quark, anti-bottom quark). [Color is not relevant for determination of mass eigenvalues, and hence effectively we have four fermions per generation: two leptons and two quarks, after suppressing color]. The trace two Jordan matrix will have entries $(2/3, 2/3, 2/3)$ and will represent the (up quark, charm, top). Lastly, the trace three Jordan matrix will have entries $(1, 1, 1)$ and will represent (positron, anti-muon, anti-tau-lepton).

We have thus identified the diagonal real entries of the four Jordan matrices whose eigenvalues we seek. We must next specify the octonionic entries in each of the four Jordan matrices. Note however that the above representation of the fermions of one generation is using complex octonions, whereas the entries in the Jordan matrices are real octonions. So we devise the following scheme for a one-to-one map from the complex octonion to a real octonion. Since we are ignoring color, we pick one out of the three up quarks, say $(e_4 + ie_5)$, and one of three anti-down quarks, say $(e_5 + ie_4)$. Since the representation for the electron and the neutrino use $e_7$ and a complex number, it follows that the four octonions we have picked form the quaternionic triplet $(e_4, e_5, e_7)$ [we use the Fano plane convention shown in the figure below]. Hence the four said octonions are in fact complex quaternions, thus belonging to the general form

\[(a_0 + ia_1) + (a_2 + ia_3)e_4 + (a_4 + ia_5)e_5 + (a_6 + ia_7)e_7\]  

where the eight $a$-s are real numbers. By definition, we map this complex quaternion to the following real octonion:

\[a_0 + a_1e_1 + a_5e_2 + a_3e_3 + a_2e_4 + a_4e_5 + a_7e_6 + a_6e_7\]  

Note that the four real coefficients in the original complex quaternion have been kept in
place, and their four imaginary counterparts have been moved to the octonion directions \((e_1, e_2, e_3, e_6)\) now as real numbers. Clearly, the map is reversible, given the real octonion we can construct the equivalent complex quaternion representing the fermion. We can now use this map and construct the following four real octonions for the neutrino, anti-down quark, up quark and the positron, respectively, after comparing with their complex octonion representation above.

\[
V_\nu = \frac{i}{2}e_7 \rightarrow \frac{1}{2}e_6
\]

(13)

\[
V_{ad} = \frac{1}{4}e_5 + \frac{i}{4}e_4 \rightarrow \frac{1}{4}e_5 + \frac{1}{4}e_3
\]

(14)

\[
V_u = \frac{1}{4}e_4 + \frac{i}{4}e_5 \rightarrow \frac{1}{4}e_4 + \frac{1}{4}e_2
\]

(15)
\[ V_{e^+} = \frac{i}{8} - \frac{1}{8} e_7 \rightarrow -\frac{1}{8} - \frac{1}{8} e_7 \]  

(16)

These four real octonions will go, one each, in the four different Jordan matrices whose eigenvalues we wish to calculate. Next, we need the real octonionic representations for the four fermions [color suppressed] in the second generation and the four in the third generation. We propose to build these as follows, from the real octonion representations made just above for the first generation. Since \( F_4 \) has the inclusion \( SU(3) \times SU(3) \), one \( SU(3) \) being for color and the other for generation, we propose to obtain the second generation by a \( 2\pi/3 \) rotation on the first generation, and the third generation by a \( 2\pi/3 \) rotation on the second generation. By this we mean the following construction, for the four respective Jordan matrices:

Up quark / Charm / Top: The up quark is \((e_4/4 + e_2/4)\) We think of this as a ‘plane’ and rotate this octonion by \(2\pi/3\) by left multiplying it by \(e^{2\pi e_2/3} = -1/2 + \sqrt{3}e_2/2\). This will be the charm quark \(V_c\). Then we left multiply the charm quark by \(e^{2\pi e_2/3}\) to get the top quark \(V_t\). Hence we have,

\[ V_c = (-1/2 + \sqrt{3}e_2/2) \times V_u = (-1/2 + \sqrt{3}e_2/2) \times \left( \frac{1}{4} e_4 + \frac{1}{4} e_2 \right) = -\frac{1}{8} e_4 - \frac{1}{8} e_2 - \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} e_1 \]

(17)

We have used the conventional multiplication rules for the octonions, which are reproduced below in Fig. 2, for ready reference. Similarly, we can construct the top quark by a \(2\pi/3\) rotation on the charm:

\[ V_t = (-1/2 + \sqrt{3}e_2/2) \times V_c = (-1/2 + \sqrt{3}e_2/2) \times \left( \frac{1}{8} e_4 - \frac{1}{8} e_2 - \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} e_1 \right) \]

(18)

\[ = -\frac{1}{8} e_4 - \frac{1}{8} e_2 + \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{8} e_1 \]

Next, we construct the anti-strange \(V_{as}\) and anti-bottom \(V_{ab}\), by left-multiplication of the anti-down quark \(V_{ad}\) by \(e^{2\pi e_3/3}\).

\[ V_{as} = \left( -\frac{1}{2} + \frac{\sqrt{3}}{2} e_3 \right) \times V_{ad} = \left( -\frac{1}{2} + \frac{\sqrt{3}}{2} e_3 \right) \times \left( \frac{1}{4} e_5 + \frac{1}{4} e_3 \right) \]

\[ = -\frac{1}{8} e_5 - \frac{1}{8} e_3 + \frac{\sqrt{3}}{8} e_2 - \frac{\sqrt{3}}{8} \]

(19)
FIG. 2. The multiplication table for two octonions. Elements in the first column on the left, left multiply elements in the top row.

\[
V_{ab} = \left( -\frac{1}{2} + \frac{\sqrt{3}}{2} e_3 \right) \left( -\frac{1}{8} e_5 - \frac{1}{8} e_3 + \frac{\sqrt{3}}{8} e_2 - \frac{\sqrt{3}}{8} \right)
\]

\[
= -\frac{1}{8} e_5 - \frac{\sqrt{3}}{8} e_2 - \frac{1}{8} e_3 + \frac{\sqrt{3}}{8}
\]

Next, we construct the octonions for the anti-muon \( V_{a\mu} \) and anti-tau-lepton \( V_{a\tau} \) by left multiplying the positron \( V_{e+} \) by \( e^{2\pi e_1/3} \)

\[
V_{a\mu} = \left( -\frac{1}{2} + \frac{\sqrt{3}}{2} e_1 \right) \times \left( -\frac{1}{8} e_1 - \frac{1}{8} e_7 \right)
\]

\[
= \frac{1}{16} e_1 + \frac{1}{16} e_7 + \frac{\sqrt{3}}{16} + \frac{\sqrt{3}}{16} e_3
\]
\[
V_{\alpha\tau} = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} e_6\right) \times \left(\frac{1}{16} e_1 + \frac{1}{16} e_7 + \frac{\sqrt{3}}{16} + \frac{\sqrt{3}}{16} e_3\right)
= \frac{1}{16} e_7 - \frac{\sqrt{3}}{16} + \frac{1}{16} e_1 - \frac{\sqrt{3}}{16} e_3
\] (22)

Lastly, we construct the octonions \(V_{\nu\mu}\) for the muon neutrino and \(V_{\nu\tau}\) for the tau neutrino, by left multiplying on the electron neutrino \(V_\nu\) with \(e^{2\pi e_6/3}\)

\[
\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} e_6\right) \times \frac{1}{2} e_6 = -\frac{1}{4} e_6 - \frac{\sqrt{3}}{4}
\] (23)

\[
V_{\nu\tau} = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} e_6\right) \times \left(-\frac{1}{4} e_6 - \frac{\sqrt{3}}{4}\right) = -\frac{1}{4} e_6 + \frac{\sqrt{3}}{4}
\] (24)

We now have all the information needed to write down the four Jordan matrices whose eigenvalues we will calculate. Diagonal entries are electric charge, and off-diagonal entries are octonions representing the particles. Using the above results we write down these four matrices explicitly. The neutrinos of three generations

\[
X_\nu = \begin{bmatrix}
0 & V_\nu & V_{\nu\mu}^* \\
V_{\nu\mu}^* & 0 & V_{\nu\tau} \\
V_{\nu\mu} & V_{\nu\tau}^* & 0
\end{bmatrix}
\] (25)

The anti-down set of quarks of three generations [anti-down, anti-strange, anti-bottom]:

\[
X_{ad} = \begin{bmatrix}
\frac{1}{3} & V_{ad} & V_{as}^* \\
V_{ad}^* & \frac{1}{3} & V_{ab} \\
V_{as} & V_{ab}^* & \frac{1}{3}
\end{bmatrix}
\] (26)

The up set of quarks for three generations [up, charm, top]

\[
X_u = \begin{bmatrix}
\frac{2}{3} & V_u & V_c^* \\
V_u^* & \frac{2}{3} & V_t \\
V_c & V_t^* & \frac{2}{3}
\end{bmatrix}
\] (27)
The positively charged leptons of three generations [positron, anti-muon, anti-tau-lepton]

\[ X_{e^+} = \begin{bmatrix} 1 & V_{e+} & V_{\mu}^* \\ V_{e+}^* & 1 & V_{\tau} \\ V_{\mu} & V_{\nu}^* & 1 \end{bmatrix} \]  \hspace{1cm} (28)

Next, the eigenvalue equation corresponding to each of these Jordan matrices can be written down, after using the expressions given above for calculating the determinant and the function \( S(X) \). Tedious but straightforward calculations with the octonion algebra give the following four cubic equations:

Neutrinos: We get \( Tr(X) = 0, S(X) = -3/4, Det(X) = 0, \) and hence the cubic equation and roots

\[ \lambda^3 - \frac{3}{4} \lambda = 0 \hspace{1cm} \text{ROOTS} : \left( -2\sqrt{\frac{3}{8}}, 0, 2\sqrt{\frac{3}{8}} \right) \]  \hspace{1cm} (29)

Anti-down-quark + its higher generations [anti-down, anti-strange, anti-bottom]: We get \( Tr(X) = 1, S(X) = -1/24, Det(X) = -19/216, \) and the following cubic equation and roots

\[ \lambda^3 - \lambda^2 - \frac{1}{24} \lambda + \frac{19}{216} = 0 \hspace{1cm} \text{ROOTS} : \frac{1}{3} - \sqrt{\frac{3}{8}}, \frac{1}{3}, \frac{1}{3} + \sqrt{\frac{3}{8}} \]  \hspace{1cm} (30)

Up quark + its higher generations [up, charm, top]: We get \( Tr(X) = 2, S(X) = 23/24, Det(X) = 5/108 \) and the following cubic equation and roots:

\[ \lambda^3 - 2\lambda^2 + \frac{23}{24} \lambda - \frac{5}{108} = 0 \hspace{1cm} \text{ROOTS} : \frac{2}{3} - \sqrt{\frac{3}{8}}, \frac{2}{3}, \frac{2}{3} + \sqrt{\frac{3}{8}} \]  \hspace{1cm} (31)

Positron + its higher generations [positron, anti-muon, anti-top-lepton]: We get \( Tr(X) = 3, S(X) = 3 - 3/32, Det(X) = 1 - 3/32 \) and the following cubic equation and roots:

\[ \lambda^3 - 3\lambda^2 + \left( 3 - \frac{3}{32} \right) \lambda - \left( 1 - \frac{3}{32} \right) = 0 \hspace{1cm} \text{ROOTS} : 1 - \frac{1}{2}\sqrt{\frac{3}{8}}, 1, 1 + \frac{1}{2}\sqrt{\frac{3}{8}} \]  \hspace{1cm} (32)
As expected from the known elementary properties of cubic equations, the sum of the roots is $Tr(X)$, their product is $Det(X)$, and the sum of their pairwise products is $S(X)$. Interestingly, this also shows that the sum of the roots is equal to the total electric charge of the three fermions under consideration in each of the respective cases. Whereas $S(X)$ and $Det(X)$ are respectively related to an invariant inner product and an invariant trilinear form constructed from the Jordan matrix, their physical interpretation in terms of fermion properties remains to be understood.

The roots exhibit a remarkable pattern. In each of the four cases, one of the three roots is equal to the corresponding electric charge, and the other two roots are placed symmetrically on both sides of the middle root, which is the one equal to the electric charge. All three roots are positive in the up quark set and in the positron set, whereas the neutrino set and anti-down quark set have one negative root each, and the neutrino also has a zero root. It is easily verified that the calculation of eigenvalues for the anti-particles yields the same set of eigenvalues, up to a sign.

One expects these roots to relate to masses of quarks and leptons for various reasons, and principally because the automorphism group of the complexified octonions contains the 4D Lorentz group as well, and the latter we know relates to gravity. Since mass is the source of gravity, we expect the Lorentz group to be involved in an essential way in any theory which predicts masses of elementary particles. And the group $F_4$, besides being related to $G_2$, and a possible candidate for the unification of the four interactions, is also the automorphism group of the EJA. We have motivated how the four projections of the EJA relate naturally to the four generation sets of the fermions. Thus there is a strong possibility that the eigenvalues of the characteristic equation of the EJA yield information about fermion mass ratios, especially it being a cubic equation with real roots. We make the following preliminary observations about the known mass ratios, in the hope that they might help give some further insight into the possible relevance of these eigenvalues.

For the set (positron, anti-muon, anti-tau-lepton), the three respective masses are known to satisfy the following empirical relation, known as the Koide formula:

$$\frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = 0.666661(7) \approx \frac{2}{3}$$ (33)
For the three roots of the corresponding cubic equation (32) we get that

\[ 2 \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{(\lambda_1 + \lambda_2 + \lambda_3)^2} = 2 \frac{[Tr(X)]^2 - 2S(X)}{[Tr(X)]^2} = \frac{2}{3} \left( 1 + \frac{1}{16} \right) \approx 0.70833 \]  

(34)

The factor 1/16 comes from the sum of the absolute values of the three octonions which go into the related Jordan matrix. This observation suggests that the eigenvalues bear some relation with the square roots of the masses of the three charged leptons, though simply comparing square roots of their mass-ratios does not seem to yield any obvious relation with the eigenvalues. Further investigation is in progress. Rather, we get the following logarithmic ratios for masses of the charged leptons [taken as 0.5 MeV, 105 MeV, 1777 Mev] and for the roots

\[ \ln \left( \frac{105}{0.5} \right)^{1/4} \approx 1.34; \quad \frac{1 + \sqrt{\frac{3}{32}}}{1} \sim 1.31 \]  

(35)

\[ \ln \left( \frac{1777}{0.5} \right)^{1/4} \approx 2.04; \quad \frac{1 + \sqrt{\frac{3}{32}}}{1 - \sqrt{\frac{3}{32}}} \sim 1.88 \]  

(36)

\[ \ln \left( \frac{1777}{105} \right)^{1/4} \approx 0.70; \quad \frac{1 + \sqrt{\frac{3}{32}}}{1 - \sqrt{\frac{3}{32}}} - \frac{1 + \sqrt{\frac{3}{32}}}{1} \sim 0.57 \]  

(37)

For the up quark set though, we see a correlation in terms of square roots of masses.

In the case of the up quark set, the following approximate match is observed between the ratios of the eigenvalues, and the mass square root ratios of the masses of up, charm and top quark. For the sake of this estimate we take these three quark masses to be [2.3, 1275, 173210] in Mev [28]. The following ratios are observed:

\[ \sqrt{\frac{1275}{2.3}} \sim 23.55; \quad \frac{2}{3} + \sqrt{\frac{3}{8}} \approx 23.56 \]  

(38)

\[ \sqrt{\frac{173210}{1275}} \sim 11.66; \quad \frac{2}{3} - \sqrt{\frac{3}{8}} \approx 12.28 \]  

(39)

\[ \sqrt{\frac{173210}{2.3}} \sim 274.42; \quad \left( \frac{2}{3} + \sqrt{\frac{3}{8}} \right) \times \left( \frac{2}{3} - \sqrt{\frac{3}{8}} \right) \approx 289.23 \]  

(40)
Within the error bars on the masses of the up set of quarks, the two sets of ratios are seen to agree with each other up to second decimal place.

Considering that one of the roots is negative in the anti-down-quark set, we have not succeeded in identifying any discernible correlation with mass ratios here. The same is true for the neutrino set, where one root is negative and one root is zero. Nonetheless, the case of the neutrino is instructive, and shows how non-zero mass could arise fundamentally, even when the electric charge is zero. In this case, the non-zero contribution comes from the inner product related quantity $S(X)$, and therein from the absolute magnitude of the octonions in the Jordan matrix, which necessarily has to be non-zero. We thus see that masses are derivative concepts, obtained from the three more fundamental entities, namely the electric charge, and the geometric invariants $S(X)$ and $Det(X)$, with the last two necessarily being defined commonly for the three generations. And since mass is the source of gravity, this picture is consistent with gravity and space-time geometry being emergent from the underlying geometry of the octonionic space which algebraically determines the properties of the elementary particles. We note that there are no free parameters in the above analysis, no dimensional quantities, and no assumption has been put by hand. Except that we identify the octonions with elementary fermions. The numbers which come out from the above analysis are number-theoretic properties of the octonion algebra.

These observations suggest a possible fundamental relation between eigenvalues of the EJA and particle masses. In the next section, we provide preliminary, modest, evidence for such a connection, based on our proposal for unification based on division algebras and a matrix-valued Lagrangian dynamics.

III. AN OCTONIONIC LAGRANGIAN FOR THE STANDARD MODEL

The action and Lagrangian for the three generations of standard model fermions, fourteen gauge bosons, and four potential Higgs bosons, are given by [24]

$$\frac{S}{C_0} = \int d\tau \mathcal{L} ; \quad \mathcal{L} = \frac{1}{2} Tr \left[ \frac{L_p^2}{L^2} \dot{Q}_1 \dot{Q}_2 \right] \quad (41)$$

Here,

$$\dot{Q}_1 = \dot{Q}_B + \frac{L_p^2}{L^2} \beta_1 \dot{Q}_F ; \quad \dot{Q}_2 = \dot{Q}_B + \frac{L_p^2}{L^2} \beta_2 \dot{Q}_F \quad (42)$$
\[ \dot{Q}_B = \frac{1}{L} (i\alpha q_B + L\dot{q}_B); \quad \dot{Q}_F = \frac{1}{L} (i\alpha q_F + L\dot{q}_F) = \] (43)

By defining
\[ q^\dagger_1 = q^\dagger_B + \frac{L^2}{L^2} \beta_1 q^\dagger_F ; \quad q_2 = q_B + \frac{L^2}{L^2} \beta_2 q_F \] (44)

we can express the Lagrangian as
\[ L = \frac{L^2}{2L^2} \text{Tr} \left[ \left( \dot{q}^\dagger_1 + \frac{i\alpha}{L} q^\dagger_1 \right) \times \left( \dot{q}_2 + \frac{i\alpha}{L} q_2 \right) \right] = \frac{L^2}{2L^2} \text{Tr} \left[ \dot{q}^\dagger_1 q_2 - \frac{\alpha^2}{L^2} \dot{q}^\dagger_1 q_2 + \frac{i\alpha}{L} \dot{q}^\dagger_1 q_2 + \frac{i\alpha}{L} \dot{q}^\dagger_1 q_2 \right] \] (45)

We now expand each of these four terms inside of the trace Lagrangian, using the definitions of \( q_1 \) and \( q_2 \) given above:
\[ \dot{q}^\dagger_1 q_2 = \dot{q}^\dagger_B q_B + \frac{L^2}{L^2} \dot{q}^\dagger_B \beta_1 q^\dagger_F q_B + \frac{L^4}{L^4} \beta_1 q^\dagger_F \beta_2 q^\dagger_F \]
\[ q^\dagger_1 q_2 = q^\dagger_B q_B + \frac{L^2}{L^2} q^\dagger_B \beta_1 q^\dagger_F q_B + \frac{L^4}{L^4} \beta_1 q^\dagger_F \beta_2 q^\dagger_F \]
\[ \dot{q}^\dagger_2 q_1 = \dot{q}^\dagger_B q_B + \frac{L^2}{L^2} q^\dagger_B \beta_2 q^\dagger_F q_B + \frac{L^4}{L^4} \beta_1 q^\dagger_F \beta_2 q^\dagger_F \]
\[ q^\dagger_2 q_1 = q^\dagger_B q_B + \frac{L^2}{L^2} q^\dagger_B \beta_2 q^\dagger_F q_B + \frac{L^4}{L^4} \beta_1 q^\dagger_F \beta_2 q^\dagger_F \] (46)

In our recent work, we suggested this Lagrangian, having the symmetry group \( F_4 \), as a candidate for unification. There are fourteen gauge bosons (equal to the number of generators of \( G_2 \)). These are the eight gluons, the three weak isospin vector bosons, the photon, and the two Lorentz bosons. These bosons, along with one Higgs, can be accounted for by the four bosonic terms which form the first column in the above four sub-equations. The remaining twelve terms were proposed to describe three fermion generations and three Higgs, with the three generations being motivated by the triality of \( SO(8) \). However, one important question which has not been addressed there is: why does triality not give rise to three copies of the bosons?! In the framework of the present approach we tentatively explore the following answer. We know that the even-grade Grassmann numbers which form the entries of the bosonic matrices are made from even-number products of odd-grade (fermionic) Grassmann numbers, and the latter are in a sense more basic. Could it then be that bosonic degrees of freedom are made from fermionic degrees of freedom? If this were to be so, it could prevent
the tripling of bosons, if we think of them as arising at the ‘intersections’ of the octonionic directions which represent fermions.

The seven imaginary unit octonions are used to make the Fano plane, which has seven points and seven lines [adding to fourteen elements; points and lines have equal status]. If we include the real direction [we have assumed $q_{B0}$ to be self-adjoint] also, we get an equivalent of a 3-D cube where the eight vertices now stand for the eight octonions, with one of them [the ‘origin’] standing for the real line. As explained by Baez: “The Fano plane is the projective plane over the 2-element field $\mathbb{Z}_2$. In other words, it consists of lines through the origin in the vector space $\mathbb{Z}_2^3$. Since every such line contains a single nonzero element, we can also think of the Fano plane as consisting of the seven nonzero elements of $\mathbb{Z}_2^3$. If we think of the origin in $\mathbb{Z}_2^3$ as corresponding to 1 in $\mathbb{O}$, we get the following picture of the octonions”. This picture is Fig. 3 below, borrowed from Baez [14]. Considering points, lines and faces together, this structure has 26 elements $[8+12+6 = 26]$. Motivated by this representation of the octonion, and the triality of $SO(8)$, we propose the following diagrammatic representation of the standard model fermions, gauge bosons, and Higgs as shown in Fig. 4. It motivates us to think of bosons as arising as ‘intersections’ of the elements representing fermions. We have taken four copies of the Baez cube, with the central one at the intersection of the other three, and used them to represent the elementary particles. We now attempt to describe Fig. 4 in some detail. There is a central black-colored cube (henceforth a cube is an octonion) in the front, which represents the fourteen gauge bosons and the four Higgs bosons; we will return to this cube shortly. Then there are three more (colored) cubes: one to the left, one at the back, and one at the bottom. These are marked as Gen I, Gen II and Gen III, and represent the three fermion generations. Let us focus first on the octonion on the left, which is Gen I, and where the eight vertices have been marked $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ just as in the Baez cube. If $e_0$ were to be excluded, this cube becomes the Fano plane [Fig. 1 above] and the arrows marked in the Gen. I cube follow the same directions as in the Fano plane. In this Gen I cube, leaving out all those elements which are at the intersection with the central bosonic cube, and leaving out the face on the far left, we are left with sixteen elements: four points, eight lines, and four faces. The four points are shown in blue and are $(e_3, e_5, e_6, e_7)$. The eight lines are: $(e_4 e_3, e_7 e_2, e_3 e_7, e_7 e_6, e_5 e_6, e_6 e_4, e_5 e_0, e_6 e_1)$. The four planes are: $(e_4 e_3 e_7 e_2), (e_0 e_5 e_6 e_1), (e_7 e_2 e_1 e_6), (e_3 e_4 e_0 e_5)$. Between them, these sixteen elements represent the eight fermions and their anti-particles in one generation, one particle
FIG. 3. The octonions [From Baez [14]].

/ anti-particle per octonionic element.

The up quark, the down quark, and their anti-particles of one particular color are (marked by) the four lines ($e_4e_3, e_7e_2, e_0e_5, e_6e_1$). The points ($e_3, e_5, e_6, e_7$) mark $u, d$ of a second color, and the lines ($e_3e_7, e_7e_6, e_3e_5, e_5e_6$) mark the $u, d$ of the third color. The four planes mark the electron, the neutrino, and their anti-particles. Between them, these sixteen elements have an $SU(3)$ symmetry: they can be correlated to the (8+8)D particle basis constructed by Furey, from the $SU(3)$ in $G_2$. Next, the Gen II and Gen III along with Gen I has another $SU(3)$ symmetry, which is responsible for the three generations. These three fermionic cubes represent three intersecting copies of $G_2$ each cube having an $SU(3)$ symmetry. The three-way intersection is $SU(2)XSU(2)$, this being the black central cube, and the bosons lie on this cube. At the same time the fermionic cubes make contact with the bosonic cube,
enabling the bosons to act on the fermions.

We now try to understand the central bosonic cube. First we count the number of its elements: it gets a total of 3x10=30 elements from the three side cubes, which when added to its own 26 elements gives a total of 56. But there are a lot of common elements, so that the actual number of independent elements is much smaller, and we enumerate them now. Three points are shared two-way and three points shared three-way and the point $e_0$ is shared four-way; that reduces the count to 44. Nine lines are shared: three of them three way, and six of them two way, reducing the count to 32. The shared three planes reduce the count to 29. We now account for the assignment of bosons to these 29 locations.

The eight gluons are on the front right, marked by the pink points, and lines labelled $g_1$ to $g_8$, and the photon is assigned to the plane $(e_3,e_7,e_6,e_5)$ on the front right enclosed by the gluons. The two Lorentz bosons are the yellow points $e_4$ and $e_1$ also marked $L_2$ and $L_1$. The
three vector bosons are marked by the lines $e_0e_1$, $e_0e_4$ and the point $e_2$, also marked $Z^0$. The Higgs $H$ is at the four way real point $e_0$. Three more Higgs are shown as follows: two planes per Higgs, e.g. the plane $e_0e_4e_2e_1$ and the mirror fermionic plane $e_3e_5e_6e_7$ on the far left in Gen I. Analogously, another Higgs is given by the bosonic plane $e_0e_1e_6e_5$ and its mirror fermionic plane at the front bottom in Gen III. The third Higgs is given by the bosonic plane $e_0e_4e_3e_5$ and its mirror fermionic plane at the back in Gen II. This way 21 elements are used up. The remaining 8 un-used elements (six lines and two planes) are assigned to eight terms in the Lagrangian representing the action of the spacetime symmetry on the gluons: these are the terms $\dot{q}_Bq_B^\dagger$ and $q_B^\dagger\dot{q}_B$ in (48).

The bosonic cube lies in the intersection of the three $G_2$ and hence does not triplicate during the $SU(3)$ rotation which generates the three fermion generations. The symmetry group of the theory is the 52 dimensional group $F_4$, with $8 \times 3 = 24$ generators coming from the three fermionic cubes, and the rest 28 from the bosonic sector $[14 + 2 \times 3 + 8 = 28]$. This diagram does suggest that one could investigate bosonic degrees of freedom as made from pairs of fermion degrees of freedom. With this tentative motivation, we return to our Lagrangian, and seek to write it explicitly as for a single generation of bosons, and three generations of fermions. Upon examination of the sub-equations in Eqn. (48) we find that the last column has terms bilinear in the fermions, and we would like to make it appear just as the second and third column do, so that we can explicitly have three fermion generations. With this intent, we propose the following assumed definitions of the bosonic degrees of freedom, by recasting the four terms in the last column of Eqn. (48):

$$
\frac{L^4}{L^1} \beta_1 \dot{q}_F^\dagger \beta_2 q_F \equiv \frac{L^2}{L^2} \dot{q}_B \beta_2 q_F + \frac{\alpha^2}{L^2} A \\
\frac{L^4}{L^3} \beta_1 \dot{q}_F^\dagger \beta_2 q_F \equiv \frac{L^2}{L^2} q_B \beta_2 q_F + A \\
\frac{L^4}{L^4} \beta_1 \dot{q}_F^\dagger \beta_2 q_F \equiv \frac{L^2}{L^2} q_B \beta_1 \dot{q}_F^\dagger + B \\
\frac{L^4}{L^4} \beta_1 \dot{q}_F^\dagger \beta_2 q_F \equiv \frac{L^2}{L^2} \dot{q}_B \beta_1 q_F^\dagger - B
$$

where $A$ and $B$ are bosonic matrices which drop out on summing the various terms to get the full Lagrangian, With this redefinition, the sub-equations Eqn. (48) can be now written
in the following form after rewriting the last column:

\[
\begin{align*}
q_1^\dagger q_2 & = q_B^\dagger q_B + \frac{L_p^2}{L^2} q_B^\dagger q_B^\beta_2 q_F + \frac{L_p^2}{L^2} \beta_1 q_F^\dagger q_B + \frac{L_p^2}{L^2} q_B^\beta_2 q_F \\
q_1^\dagger q_2 & = q_B^\dagger q_B + \frac{L_p^2}{L^2} q_B^\dagger q_B^\beta_2 q_F + \frac{L_p^2}{L^2} \beta_1 q_F^\dagger q_B + \frac{L_p^2}{L^2} q_B^\beta_2 q_F \\
q_1^\dagger q_2 & = q_B^\dagger q_B + \frac{L_p^2}{L^2} q_B^\dagger q_B^\beta_2 q_F + \frac{L_p^2}{L^2} \beta_1 q_F^\dagger q_B + \frac{L_p^2}{L^2} q_B^\beta_2 q_F \\
q_1^\dagger q_2 & = q_B^\dagger q_B + \frac{L_p^2}{L^2} q_B^\dagger q_B^\beta_2 q_F + \frac{L_p^2}{L^2} \beta_1 q_F^\dagger q_B + \frac{L_p^2}{L^2} q_B^\beta_2 q_F
\end{align*}
\]

The terms now look harmonious and we can see a structure emerging - the first column are bosonic terms and these are not triples. The remaining terms are four sets of three each [to which their adjoints will eventually get added] which can clearly describe three generations of the four sets, which is what we had in the Jordan matrices in the previous section. Putting it all together, we can now rewrite the Lagrangian so that it explicitly looks like the one for gauge bosons and four sets of three generations of fermions, as in the Jordan matrix:

\[
\mathcal{L} = \frac{L_p^2}{2L^2} \text{Tr} \left[ \left( \tilde{q}_1^\dagger + i\alpha \frac{\dot{q}_1}{L} \right) \times \left( \tilde{q}_2^\dagger + i\alpha \frac{\dot{q}_2}{L} \right) \right] \\
= \frac{L_p^2}{2L^2} \text{Tr} \left[ \tilde{q}_2^\dagger \dot{q}_2 - \alpha^2 \frac{\dot{q}_1}{L} \tilde{q}_1^\dagger q_2 + i\alpha \frac{\dot{q}_1}{L} \tilde{q}_1^\dagger q_2 + i\alpha \frac{\dot{q}_1}{L} \tilde{q}_1^\dagger q_2 \right] \\
\equiv \frac{L_p^2}{2L^2} \text{Tr} \left[ \mathcal{L}_{\text{bosons}} + \mathcal{L}_{\text{set1}} + \mathcal{L}_{\text{set2}} + \mathcal{L}_{\text{set3}} + \mathcal{L}_{\text{set4}} \right]
\]

where

\[
\mathcal{L}_{\text{bosons}} = \tilde{q}_B^\dagger \dot{q}_B - \alpha^2 \frac{\dot{q}_B}{L} \tilde{q}_B^\dagger q_B + i\alpha \frac{\dot{q}_B}{L} \tilde{q}_B^\dagger q_B + i\alpha \frac{\dot{q}_B}{L} \tilde{q}_B^\dagger q_B
\]

\[
\mathcal{L}_{\text{set1}} = \frac{L_p^2}{L^2} \tilde{q}_B^\dagger \beta_2 q_F + \frac{L_p^2}{L^2} \beta_1 \tilde{q}_F^\dagger q_B + \frac{L_p^2}{L^2} \tilde{q}_B^\beta_2 q_F
\]

\[
\mathcal{L}_{\text{set2}} = -\alpha^2 \left( \frac{L_p^2}{L^2} \tilde{q}_B^\dagger \beta_2 q_F + \frac{L_p^2}{L^2} \beta_1 \tilde{q}_F^\dagger q_B + \frac{L_p^2}{L^2} \tilde{q}_B^\beta_2 q_F \right)
\]

\[
\mathcal{L}_{\text{set3}} = \frac{i\alpha}{L} \left( \frac{L_p^2}{L^2} \tilde{q}_B^\dagger \beta_2 q_F + \frac{L_p^2}{L^2} \beta_1 \tilde{q}_F^\dagger q_B + \frac{L_p^2}{L^2} \tilde{q}_B^\beta_2 q_F \right)
\]

\[
\mathcal{L}_{\text{set4}} = \frac{i\alpha}{L} \left( \frac{L_p^2}{L^2} \tilde{q}_B^\dagger \beta_2 q_F + \frac{L_p^2}{L^2} \beta_1 \tilde{q}_F^\dagger q_B + \frac{L_p^2}{L^2} \tilde{q}_B^\beta_2 q_F \right)
\]

We see that each of these four fermionic sets could possibly be related to a Jordan matrix, after including the adjoint part. We also see that different coupling constants appear in
different sets with identical coupling in third and fourth set and no coupling in the first set.
The first set could possibly describe neutrinos (only gravitational and weak interaction),
the second set charged leptons, and the third and fourth set the quarks. To establish this,
equations of motion remain to be worked out and then related to the eigenvalue problem.
As noted earlier, $L$ relates to mass, and this approach could reveal how the eigenvalues of
the EJA characteristic equation relate to mass. This investigation is currently in progress.

IV. CONCLUDING REMARKS

We have not addressed the question as to how these discrete order one eigenvalues might
relate to actual low values of fermion masses, which are much lower than Planck mass.
We speculatively suggest the following scenario, which needs to be explored further. The
universe is eight-dimensional, not four. The other four internal dimensions are not compactified; rather the universe is very ‘thin’ in those dimensions but they are expanding as well. There are reasons having to do with the so-called Karolyhazy uncertainty relation [29],
because of which the universe expands in the internal dimensions at one-third the rate, on
the logarithmic scale, compared to our 3D space. That is, if the 4D scale factor is $a(\tau)$,
the internal scale factor is $a_{int}^{1/3}(\tau)$, in Planck length units. Taking the size of the observed
universe to be about $10^{61}$ Planck units, the internal dimensions have a width approximately
$10^{20}$ Planck units, which is about $10^{-13}$ cm, thus being in the quantum domain. Classical
systems have an internal dimension width much smaller than Planck length, and hence they
effectively stay in [and appear to live in] four dimensional space-time. Quantum systems
probe all eight dimensions, and hence live in an octonionic universe.

The universe began in a unified phase, via an inflationary 8D expansion possibly resulting
as the aftermath of a huge spontaneous localisation event in a ‘sea of atoms of space-time-matter’ [30]. The mass values are set, presumably in Planck scale, at order one values
dictated by the eigenvalues reported in the present paper. Cosmic inflation scales down
these mass values at the rate $a^{1/3}(\tau)$, where $a(\tau)$ is the 4D expansion rate. Inflation ends
after about sixty e-folds, because seeding of classical structures breaks the color-electro-weak-
Lorentz symmetry, and classical spacetime emerges as a broken Lorentz symmetry. The
electro-weak symmetry breaking is actually a electro-weak Lorentz symmetry breaking, which
is responsible for the emergence of gravity, weak interaction being its short distance limit.
There is no reheating after inflation; rather inflation resets the Planck scale in the vicinity of the electro-weak scale, and the observed low fermion mass values result. The electro-weak symmetry breaking is mediated by the Lorentz symmetry, in a manner consistent with the conventional Higgs mechanism. It is not clear why inflation should end specifically at the electro-weak scale: this is likely dictated by when spontaneous localisation becomes significant enough for classical spacetime to emerge. It is a competition between the strength of the electro-colour interaction which attempts to bind the fermions, and the inflationary expansion which opposes this binding. Eventually, the expanding universe cools enough for spontaneous localisation to win, so that the Lorentz symmetry is broken. It remains to prove from first principles that this happens at around the electro-weak scale and also to investigate the possibly important role that Planck mass primordial black holes might play in the emergence of classical spacetime. I would like to thank Roberto Onofrio for correspondence which has influenced these ideas. See also [31].

Additional internal spatial dimensions which are not compact, yet very thin, offer a promising resolution to the quantum non-locality puzzle, thereby lifting the tension with 4D special relativity. Let us consider once again Baez’s cube of Fig. 3. Any of the three quaternionic spaces containing the unit element 1 can play the role of the emergent 4D classical space-time in which classical systems evolve. Let us say this classical universe is the plane \((1e_0e_1e_5)\). Now, the true universe is the full 8D octonionic universe, with the four internal dimensions being probed [only by] quantum systems. Now we must recall that these four internal dimensions are extremely thin, of the order of Fermi dimensions, and along these directions no point is too far from each other, even if their separation in the classical 4D quaternion plane is billions of light years! Consider then, that Alice at 1 and Bob at \(e_1\) are doing space-like separated measurements on a quantum correlated pair. Whereas the event at \(e_1\) is outside the light cone of 1, the correlated pair is always within each other’s quantum wavelength along the internal directions, say the path \((1e_3e_2e_7e_1)\). The pair influences each other along this path acausally, because this route is outside the domain of 4D Lorentzian spacetime and its causal light-cone structure. The internal route is classically forbidden but allowed in quantum mechanics. This way neither special relativity nor quantum mechanics needs to be modified. It is also interesting to ask if evolution in Connes time in this 8D octonionic universe obeying generalised trace dynamics can violate the Tsirelson bound.

Lastly we mention that the Lagrangian (45) that we have been studying closely resembles
the Bateman oscillator [32] model, for which the Lagrangian is

\[ L = m \dot{x} \dot{y} + \gamma (x \dot{y} - \dot{x} y) - kxy \quad (55) \]

I thank Partha Nandi for bringing this fact to my attention. Considering that the Bateman oscillator represents a double oscillator with relative opposite signs of energy for the two oscillators undergoing damping, it is important to understand the implications for our theory. In particular, could this imply a cancellation of zero point energies between bosonic and fermionic modes, thus annulling the cosmological constant? And also whether this damping is playing any possible role in generating matter-anti-matter asymmetry?

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